Lecture 13

Let $A(\beta)$ be a $n\times n$ real analytic self-adjoint family of matrices for β in some region $\Omega \subset \mathbb{C}$.

We have shown that there are s functions $\lambda_1(\beta), \ldots, \lambda_s(\beta)$, $1 \le s \le n$, that represent the eigenvalues of $A(\beta)$. Away from a discrete set $D \subset \Omega$, the $\lambda_i(\beta)$'s are analytic, and at points d of D, the $\lambda_i(\beta)$'s have at worst algebraic singularities. At real points $d \in D$, the $\lambda_i(\beta)$'s are analytic: Hence the $\lambda_i(\beta)$'s are real analytic functions on $L = \Omega \cap \mathbb{R}$. For $\beta \in \Omega \setminus D$, $\lambda_i(\beta) \neq \lambda_j(\beta)$ for $i \neq j$. But for $\beta \in D$, $\lambda_i(\beta) = \lambda_j(\beta)$ for some $i \neq j$. In other words the points of D where $A(\beta)$ has eigenvalues with multiplicity > 1.

Now, what about the eigenvectors of $A(\beta)$?

Consider, for example, $A(\beta) = \begin{pmatrix} 0 & \beta \\ \beta & 1+\beta \end{pmatrix}$ which has eigenvalues $\lambda_\pm(\beta) = \frac{1}{2}\left(1+\beta \pm \sqrt{(1+\beta)^2 + 4\beta^2}\right)$. The eigenvalues are simple as long as $\beta \neq 1/(\pm 2i - 1)$. In particular $D \cap \mathbb{R} = \emptyset$ and $\lambda_\pm(\beta)$ are real analytic on $\mathbb{R}$, as they should be, and the normalized eigenvectors are

$$v_\pm(\beta) = \frac{1}{\sqrt{\beta^2 + \lambda_\pm^2(\beta)}} \begin{pmatrix} \beta \\ \lambda_\pm(\beta) \end{pmatrix}$$

which are real analytic, normalized and orthogonal.

Consider another example, $A(\beta) = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}$ with

eigenvalues $\lambda_\pm(\beta) = \pm\beta$ and associated eigenvectors (in fact, constant here)

$v_\pm(\beta) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$ which are again real analytic, normalized

and orthogonal. Note that the projections onto the

eigenspaces for $\lambda_\pm(\beta)$ are given by

$$P_\pm(\beta) = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}, \quad \beta \in \mathbb{R} \setminus \{0\}$$

Here $D = \{\beta = 0\}$. It is important to note that not only are $\lambda_\pm(\beta)$ analytic at $\beta = 0$, as they should be, but also $P_\pm(\beta)$ are analytic $\forall \beta \in \mathbb{R}$. In particular

$$P_\pm(0) = \lim_{\beta \to 0} P_\pm(\beta)$$

exist. But $P_\pm(0)$ is no longer the projection

onto the eigenspace for $\lambda_+(0) = 0$: the projection onto the eigenspace for $\lambda_+(0) = 0$ is $P_+(0) + P_-(0) = I$, as $\lambda = 0$ has multiplicity 2. The remarkable fact is that these properties are preserved in the general $n \times n$ self-adjoint case.

We shall prove the following theorem

<u>Theorem 180.1</u>

Let $A(\beta)$ be a $n \times n$ real analytic self adjoint family (of matrices) for β in some region $\Omega \subset \mathbb{C}$. Let $L = \Omega \cap \mathbb{R}$.

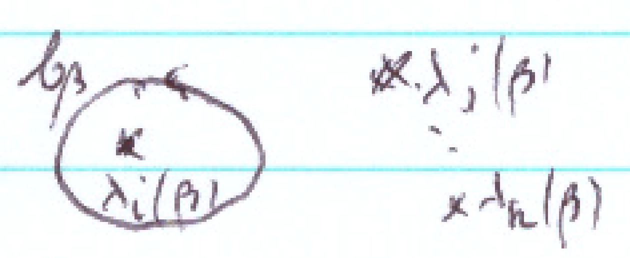
Then the eigenvalues of $A(\beta)$ are real analytic functions of $\beta \in L$ and the eigenvectors of $A(\beta)$ can be chosen to form a real analytic orthonormal basis for $\mathbb{C}^n$.

For any i, $1 \le i \le s$, let $P_i(\beta)$ be the projection onto the eigenspace associated with $\lambda_i(\beta)$. Then

(180.2) $$P_i(\beta) = \frac{1}{2\pi i} \oint_{C_\beta} \frac{1}{z - A(\beta)} \, dz$$

where

C_β is a (small) circle centered at $\lambda_i(\beta)$ and all the other eigenvalues $\lambda_j(\beta)$ in its exterior



Now each $\lambda_i(\beta)$ is in general a branch of a multi-valued analytic function, but for $\beta \in \mathbb{R}$, and hence for β in a (narrow) strip about $\mathbb{R} \cap \Omega$, the $\lambda_i(\beta)$ are single valued analytic functions. Hence $P_i(\beta)$ is a well defined analytic function for β in the strip, provided $\beta \notin D$. How does $P_i(\beta)$ behave as $\beta \to d \in D \cap \mathbb{R}$?

As $\beta \to d$, we have $\lim_{\beta \to d} \lambda_i(\beta) = \lim_{\beta \to d} \lambda_j(\beta) \equiv \hat{\lambda}$

for some $j \neq i$ and so we have to make b_β

small and smaller as $\beta \to d$ in order to exclude $\lambda_j(\beta)$

from the interior of b_β. But as the integrand in

(180.2) is given using Cramers rule by $\dfrac{q(\beta, \mu)}{\prod_{i=1}^{s} (\beta - \lambda_i(\beta))^{m_i}}$

(here m_i is the algebraic multiplicity of λ_i), $q(\beta, \mu)$ analytic,

it is clear that $P_i(\beta)$ could blow up as $|\beta - \lambda_i(\beta)| \to 0$.

The following result is basic.

<u>Lemma 181.1</u> (see Kato pp 70-71)

$P_i(\beta)$ is analytic in the punctured disc $\{0 < |\beta - d| < \varepsilon\}$

ε sufficiently small, and its Laurent expansion has only finitely many negative powers

(181.2) $$P_i(\beta) = \frac{A_{-k}}{(\beta - d)^k} + \dots + \frac{A_{-1}}{(\beta - d)} + A_0 + A_1(\beta - d) + \dots$$

for some $0 \le k < \infty$.

Proof: From (180.2), we have

$$\|P_i(\beta)\| \le 2\pi \max_{z \in \mathcal{C}_\beta} \left\| \frac{1}{z - A(\beta)} \right\|$$

But as noted above $\frac{1}{z - A(\beta)} = \frac{d(z,\beta)}{\prod_i^s (z - \lambda_i(\beta))^{m_i}}$

and so for $z \in \mathcal{C}_\beta$, $\beta \to d$

(183.1) $$\left\| \frac{1}{z - A(\beta)} \right\| \le \frac{c}{\prod_i^s |z - \lambda_i(\beta)|^{m_i}}$$

for some constant $c < \infty$. Now each of the λ_i's is real analytic a $\beta = d$ and so $|\lambda_j(\beta) - \lambda_i(\beta)| \ge c|\beta - d|^p$ for some $p \ge 1$ for all $j \ne i$. So we choose the radius r of $\mathcal{C}_\beta$ to be $|\beta - d|^q$, where $q > p$. This ensures that all $\lambda_j(\beta)$, $j \ne i$, lie outside of $\mathcal{C}_\beta$. It follows from (180.2) and (183.1) that as $\beta \to d$

(183.2) $$\|P_i(\beta)\| \le c\,|\beta - d|^{-(n-1)q}$$

Now $P_i(\beta)$ is clearly analytic in the punctured disc and (181.2) follows from (183.2) and the general theory of Laurent series. □.

Remark 183.1

Every function $f(z)$ analytic in a punctured disk $\{0 < |z| < \rho\}$ and a Laurent series uniformly convergent in the punctured disk. In general $f(z)$ will have an infinite number of negative powers, e.g.,

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots \quad , \quad z \neq 0$$

Now as $z \to 0$, $e^{1/z}$ blows up as $z \downarrow 0$, z real, but goes to 0 as $z \uparrow 0$, z real. Also if $z \to 0$ along the imaginary axis, then $e^{1/z}$ oscillates and remains bounded. The significance of the situation where $f(z)$ has only a finite number of positive powers

$$f(z) = \frac{a_{-k}}{z^k} + \cdots + \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots \quad , \quad k \geq 1$$

$a_{-k} \neq 0$, is that $f(z)$ blows up no matter how $z \to 0$.

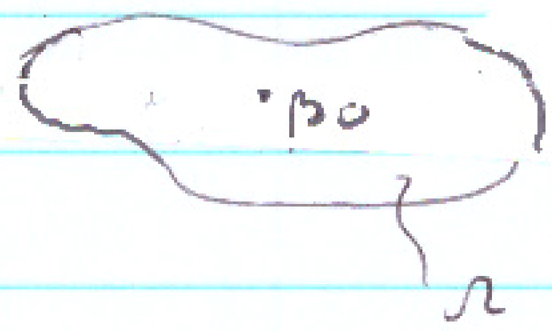
Corollary to Lemma 181.1

For a self-adjoint family $A(\beta)$, $P_i(\beta)$ is analytic on $L = \Omega \cap \mathbb{R}$.

<u>Proof:</u> For β real, $\beta \notin D$, $P_i(\beta) = \frac{1}{2\pi i}\int_{\mathcal{C}_\beta} \frac{dz}{z - A(\beta)}$ is self-adjoint. Hence $\|P_i(\beta)\| = 1$. But then by (181.2) and Remark 183.1, we must have $k=0$, and so $P_i(\beta)$ is an analytic function on $\mathbb{R}$. $\square$.

<u>Remark</u> More precisely $P_i(\beta)$ extends from $L \setminus D$ to the points d in $D \cap \mathbb{R}$, so that $P_i(\beta)$ becomes analytic on L. Note $P_i(d) = \lim_{\beta \to d} P_i(\beta)$ is not the projection associated with $\lambda_i(d)$: cf the example $A(\beta) = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}$ on p 179.

<u>Theorem 184.1</u> (see Kato pp 104-106)
Suppose $A(\beta)$ is an $n \times n$ analytic self-adjoint family in a region $\Omega \subset \mathbb{C}$ and suppose $\beta_0 \in \Omega$.



Assume that for all $i \in \{1, \ldots, s\}$ $P_i(\beta)$, the projections onto the eigenspaces associated with $\lambda_i(\beta)$, are analytic in Ω. Then there exists an analytic invertible matrix $U(\beta)$ such that

$$P_i(\beta) = U(\beta)\, P_i(\beta_0)\, U(\beta)^{-1} \quad \text{for all } \beta \in \Omega. \tag{184.2}$$

Proof We use the fact that

(185.1) $$P_i(\beta) P_j(\beta) = \delta_{ij} P_i(\beta)$$

which follows from the definitions of the $P_i(\beta)$'s. Also

(185.2) $$\sum_{i=1}^{s} P_i(\beta) = \mathbb{1}$$

(185.3) Let $$Q(\beta) = \frac{1}{2} \sum_{i=1}^{s} [P_i'(\beta), P_i(\beta)]$$

where $[A, B] = AB - BA$ and $P_i'(\beta) = \frac{d}{d\beta} P_i(\beta)$

As $P_i^2(\beta) = P_i(\beta)$ we have $P_i' P_i + P_i P_i' = P_i'$

Thus $$\sum_{i=1}^{s} P_i' P_i + \sum_{i=1}^{s} P_i P_i' = \sum_1^s P_i' = \frac{d}{ds} I = 0$$

Hence

(185.4) $$Q(\beta) = \sum_{i=1}^{s} P_i'(\beta) P_i(\beta) = - \sum_{i=1}^{s} P_i(\beta) P_i'(\beta)$$

Let $X(\beta)$ solve

(185.5) $$X'(\beta) = Q(\beta) X(\beta)$$

with initial condition $X(\beta_0)$

The solution $X(\beta)$ exists and is unique for all $\beta \in \Omega$

by the standard theory of linear ordinary differential equations.

(185.6) Now $P_i'(\beta) = [Q(\beta), P_i(\beta)]$ $\forall i = 1, \dots, s.$

Indeed from (185.1) we have

$$P_i' P_j + P_i P_j' = \delta_{ij} P_i'$$

$$\Rightarrow \quad P_i P_i' P_j + P_i P_j' = \delta_{ij} P_i P_i' = P_j P_i P_i'$$

$$\Rightarrow \quad -[P_i P_i', P_j] = -P_i P_i' P_j + P_j P_i P_i' = P_i P_j'$$

Summing over i yields (185.6), by virtue of (185.2) and (185.4)

Let $U(\beta)$ solve (185.5)

(186.1) $$U' = Q(\beta) U$$

with $U(\beta_0) = 1$. Then clearly $X(\beta) = U(\beta) X(\beta_0)$

Now $(P_i U)' = P_i' U + P_i U'$
$= (P_i' + P_i Q) U$
$= Q(P_i U)$, by (185.6)

Now $U(\beta) P_i(\beta_0)$ and $P_i(\beta) U(\beta)$

clearly solve the same differential equation as

$$U(\beta_0) P_i(\beta_0) = P_i(\beta_0) \quad (= X(\beta_0)).$$
and $$P_i(\beta_0) U(\beta_0) = P_i(\beta_0)$$

Hence by uniqueness

(186.2) $$U(\beta) P_i(\beta_0) = P_i(\beta) U(\beta)$$

Let $V(\beta)$ solve

$$V'(\beta) = -V(\beta) Q(\beta), \quad V(\beta_0) = 1$$

Then $(VU)' = V'U + VU' = -VQU + VQU = 0$

Thus $V(\beta) U(\beta) = \text{const} = I$

Thus $U(\beta)^{-1}$ exists, and equals $V(\beta)$

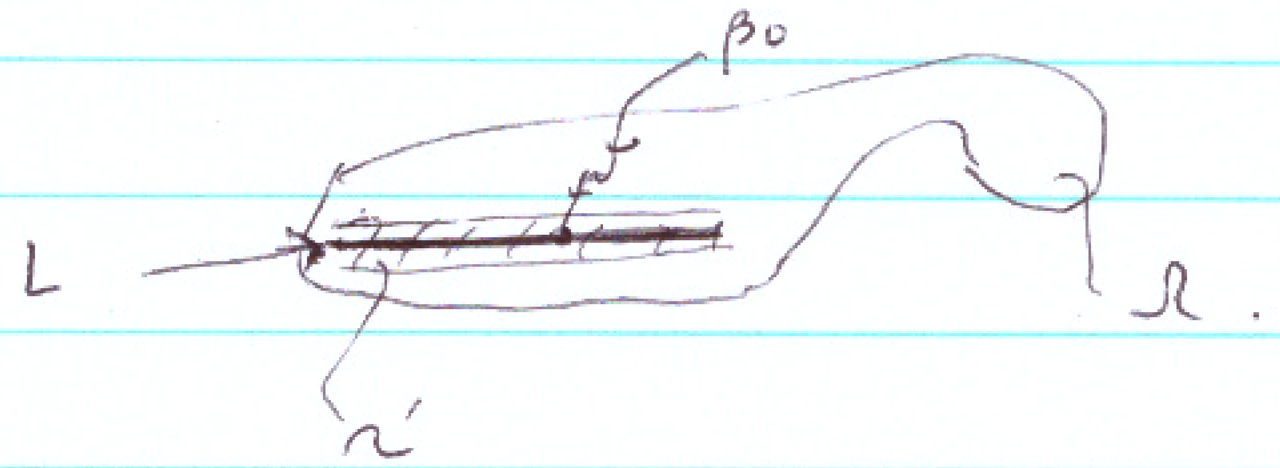
The desired result (184.2) now follows from (186.2). □

Finally we can prove Theorem 180.1

Proof: It is enough to consider that L is a real interval in Ω. If $L = \Omega \cap \mathbb{R}$ is not connected

we proceed component by component.

Without loss we can restrict our considerations to a thin strip $\Omega' \subset \Omega$ containing L



Fix $\beta_0 \in L$.

Then by our previous results $P_i(\beta)$ is analytic in Ω' and we can apply Thm (184.1) to conclude that for $\beta \in L \subset \mathbb{R}$ in particular

$$P_i(\beta) = U(\beta)\, P_i(\beta_0)\, U(\beta)^{-1}$$

Now U solve $U' = QU$, $U(\beta_0) = \mathbb{1}$.
where $Q(\beta) = \frac{1}{2} \sum_{i=1}^{s} [P_i'(\beta), P_i(\beta)]$. Now for β real,

$P_i(\beta)$ is self-adjoint (check this). Hence $P_i'(\beta)$ is self-adjoint

and so $Q^* = \frac{1}{2}\sum_1^s [P_i^*, P_i'^*] = \frac{1}{2}\sum_1^s [P_i, P_i'] = -Q$

Hence $(U^*)' = U^* Q^* = -U^* Q$

and it follows that $(U^* U)' = (U^*)' U + U^* U'$
$= -U^* Q U + U^* Q U = 0$

Hence $U^*(\beta) U(\beta) = \text{constant} = 1$

So $U(\beta)$ is unitary and $U^{-1}(\beta) = U^*(\beta)$

Using the relation $P_i(\beta) U(\beta) = U(\beta) P_i(\beta_0)$

Fixing an orthonormal bas $u_{i1}, \ldots, u_{im_i}$ for Ran $P_i(\beta_0)$
(m_i = multiplicity of $\lambda_i(\beta_0)$)
we obtain an analytic orthonormal basis $u_{ij}(\beta) = U(\beta) u_{ij}$,

$j = 1, \ldots, m_i$ for the eigenspace Ran $P_i(\beta)$ of $A(\beta)$, $\beta \in L$.

Constructing similar orthonormal sets for all j, we

obtain the desired real analytic orthonormal basis for $A(\beta)$ on L. □

Remark It is important to know what Theorem 180.1 does not say. (Consider $A(\beta) = A_0 + \beta B$, A_0 & B Hermitian.) If λ_0 is a degenerate eigenvalue of A_0 (multiplicity $= m$)

and if $w_0 \neq 0$ is an eigenvector of A_0, $(A_0 - \lambda_0) w_0 = 0$,

the theorem does not say that $\exists$ an analytic eigenvector $w(\beta)$

such that $(A(\beta) - \lambda(\beta)) w(\beta) = 0$, $\lambda(0) = \lambda_0$, where $w(0) = w_0$

ie. not every vector in the null space of $A_0 - \lambda_0$ continues

analytically to $\beta \neq 0$.

For example, consider $A(\beta) = \begin{pmatrix} 1+\beta & 0 \\ 0 & 1-\beta \end{pmatrix}$. A_0 has an eigenvalue $\lambda_0 = 1$ of multiplicity 2. The eigenvalues of $A(\beta)$ are $\lambda_1(\beta) = 1+\beta$, $\lambda_2(\beta) = 1-\beta$ with associated eigenvectors $v_1(\beta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2(\beta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, for $\beta \neq 0$. Thus (essentially, apart from

scaling) the only eigenvectors of $\mathrm{Nul}(A(0) - \lambda_0)$ that continue analytically are

$$v_1(0) = \lim_{\beta \to 0} v_1(\beta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v_2(0) = \lim_{\beta \to 0} v_2(\beta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In the case $A(\beta) = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} = 0 + \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we again have degeneracy at $\beta = 0$, but now the eigenvectors that continue are

$$v_1(0) = \lim_{\beta \to 0} v_1(\beta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2(0) = \lim_{\beta \to 0} v_2(\beta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Thus the degeneracy at $\beta = 0$ is resolved by the perturbation. This is a very general phenomenon in the perturbation of singular/degenerate problems.

Finally we can ask what role does analyticity play in perturbation Theory? If $A(\beta)$ is no longer analytic in β, the general rule is that the eigenvalues will behave "reasonably". The eigenvectors, less so.

The following is Theorem 6.8 on p122 in Kato:

Th^m. Assume $A(\beta)$ is $n \times n$, self-adjoint and continuously differentiable on an interval $I \subset \mathbb{R}$ of β. Then there exist n continuously differentiable functions $\mu_i(\beta)$ on I that represent the eigenvalues of $A(\beta)$

But with eigenvectors, everything can break down. For example consider the self-adjoint matrices

$$A(\beta) = e^{-1/\beta^2} \begin{pmatrix} \cos 2/\beta & \sin 2/\beta \\ \sin \frac{2}{\beta} & -\cos 2/\beta \end{pmatrix}, \quad \beta \neq 0$$

$A(0) = 0$

Then $A(\beta)$ is infinitely differentiable, but not analytic, on $\mathbb{R}$ and one verifies easily that the eigenvalues are $\pm e^{-2/\beta^2}$, which are clearly also infinitely differentiable. But the associated spectral projections are

$$\begin{pmatrix} \cos^2(\frac{1}{\beta}) & \cos(\frac{1}{\beta})\sin(\frac{1}{\beta}) \\ \cos(\frac{1}{\beta})\sin(\frac{1}{\beta}) & \sin^2(\frac{1}{\beta}) \end{pmatrix} \text{ and } \begin{pmatrix} \sin^2(\frac{1}{\beta}) & -\cos(\frac{1}{\beta})\sin(\frac{1}{\beta}) \\ -\cos(\frac{1}{\beta})\sin(\frac{1}{\beta}) & \cos^2(\frac{1}{\beta}) \end{pmatrix}$$

and one sees that no eigenvectors of $A(\beta)$ are even continuous at $\beta = 0$.

Regarding matrices which depend on more than one parameter, $A = A(\beta_1, \beta_2, \dots)$ very little is know beyound the perturbation theory of simple eigenvalues discussed earlier. For example for $A(\beta_1, \beta_2) = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & -\beta_1 \end{pmatrix}$, the eigenvalues are $\lambda = \pm\sqrt{\beta_1^2 + \beta_2^2}$ which are not analytic near $(\beta_1, \beta_2) = (0,0)$, even thout $A(\beta_1, \beta_2)$ is self-adjoint for $\beta_1, \beta_2 \in \mathbb{R}$.