Lecture 5 Linear Algebra Fall 2019

Functional Calculus

By a <u>functional calculus</u> we mean, loosely speaking, a rule for associating with a functional $f: \mathbb{C} \to \mathbb{C}$ and a matrix A, another matrix, written $f(A)$, in such a way as to preserve the algebra of functions i.e.

(i) $(\lambda f)(A) = \lambda (f(A))$
(ii) $(f+g)(A) = f(A) + g(A)$
(iii) $(fg)(A) = f(A)\, g(A)$

It is natural to require in addition to (i) (ii) (iii), the following:

(iv) $1(A) = I$, and
(v) $f(A) = A$ where $f(z) = z$.

Such a calculus, were it to exist, would clearly be very useful: e.g. consider the equation $Ax = b$. Then if we denote the map $z \mapsto 1/z$ by f, and if we could form $f(A)$, then clearly,

$$
\begin{aligned}
f(A)\,A &= f(A)\,g(A) \qquad , \text{ where } g(z) = z \\
&= (fg)(A) \\
&= 1(A) \\
&= I
\end{aligned}
$$

Hence

$$x = f(A)\,A\,x = f(A)\,b$$

In other words, the inverse of A, A^{-1}, can be thought of as a special case of the functional calculus.

If $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, a polynomial, then it follows from (i) $\cdots$ (v) that

(63.0) $$f(A) = a_0 + a_1 A + \cdots + a_n A^n .$$

But what about more general functions f? If f is entire, so that $f(z)$ has a convergent power series expansion, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in \mathbb{C}$, then we may define

(63.1) $$f(A) \equiv \sum_{n=0}^{\infty} a_n A^n$$

for any matrix A. Note that $f(A)$ in (63.1) is well-defined. Indeed, consider the partial sums

$$f_n(A) = \sum_{i=0}^{n} a_i A^i$$

Then for $n > m$,

$$\| f_n(A) - f_m(A) \| = \| \sum_{k=m+1}^{n} a_k A^k \|$$

$$\leq \sum_{k=m+1}^{n} |a_k| \, \|A^k\|$$

$$\leq \sum_{k=m+1}^{n} |a_k| \, \|A\|^k$$ (note: as usual $\|\cdot\|$ denotes the operator norm).

and it follows from the convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ for $f(z)$ for all z, that $\{f_n\}$ is a Cauchy sequence and hence $\lim_{n \to \infty} f_n(A)$ exist. This limit is what we mean by $f(A)$ in (63.1).

<u>Exercise</u> (63.1) defines a functorial calculus for entire functions $f(z)$ for all A. Indeed, verify (i)..(v) above.

Such a functorial calculus for entire f's is very useful.

For example, if $f_t(z) = e^{tz}$, we can form

$$f_t(A) = e^{tA} \equiv \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i$$

This function allows us to solve the differential equation

(64.1) $$du/dt = Au, \quad u(0) = a$$

Indeed, consider the vector

$$v(t) = f_t(A)a = e^{tA}a$$

We have by (i)..(v)

$$\frac{v(t+h)-v(t)}{h} = \frac{(f_{t+h}(A) - f_t(A))a}{h}$$

$$= \frac{(f_h(A)f_t(A) - f_t(A))a}{h}$$

$$(\text{as } e^{tz}e^{hz} = e^{(t+h)z})$$

$$= \frac{(f_h(A) - I)}{h} f_t(A)a$$

$$= \left[\left(A + \sum_{i=2}^{\infty} \frac{h^{i-1}}{i!} A^i\right)\right] f_t(A)a$$

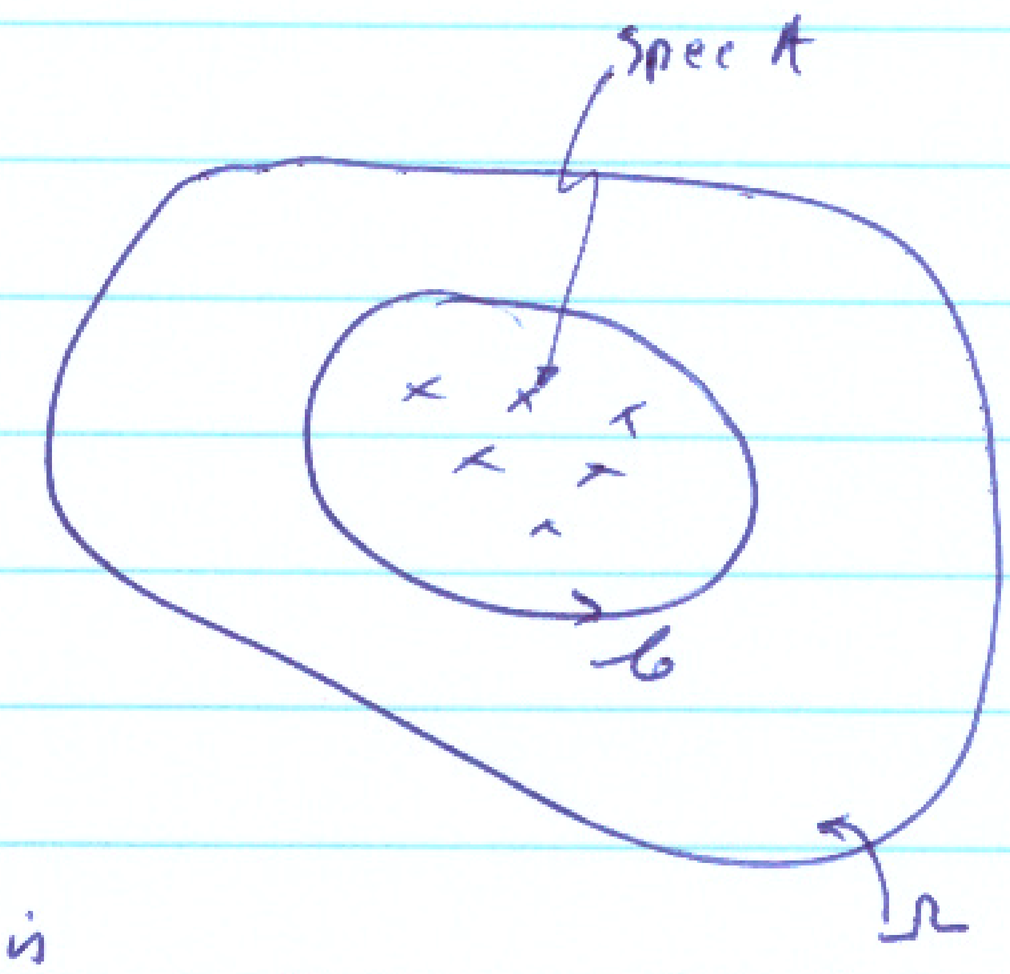
$$\to A\,v(t), \text{ as } h \to 0.$$

ie $dv/dt = Av$ and $v(0) = e^{0A}a = Ia = a$. Thus.

$v_t = e^{tA}a$ is the unique (why?) solution of the differential equation (64.1). However this functional calculus is too limited. For example, as $\sqrt{z}$ and $\log z$ are not entire, we are unable to define $\sqrt{A}$ and $\log A$ by this method.

We make the following definition which associates to all matrices A with spectrum contained in a fixed simply connected region $\Omega \subset \mathbb{C}$, a matrix $f(A)$ whenever f is analytic in Ω. Of course, if $\Omega = \mathbb{C}$, then $f(A)$ will/must coincide with (63.1) (cf, e.g. (72.2) below). (Recall a region Ω in $\mathbb{C}$ is an open connected set in $\mathbb{C}$.)

So suppose f is analytic in Ω and $\operatorname{spec}(A) \subset \Omega$. Choose a positively oriented contour $\mathcal{C} \subset \Omega$ enclosing the spectrum of A and set



(66.1) $$f(A) \equiv \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(z)}{z-A}\,dz.$$

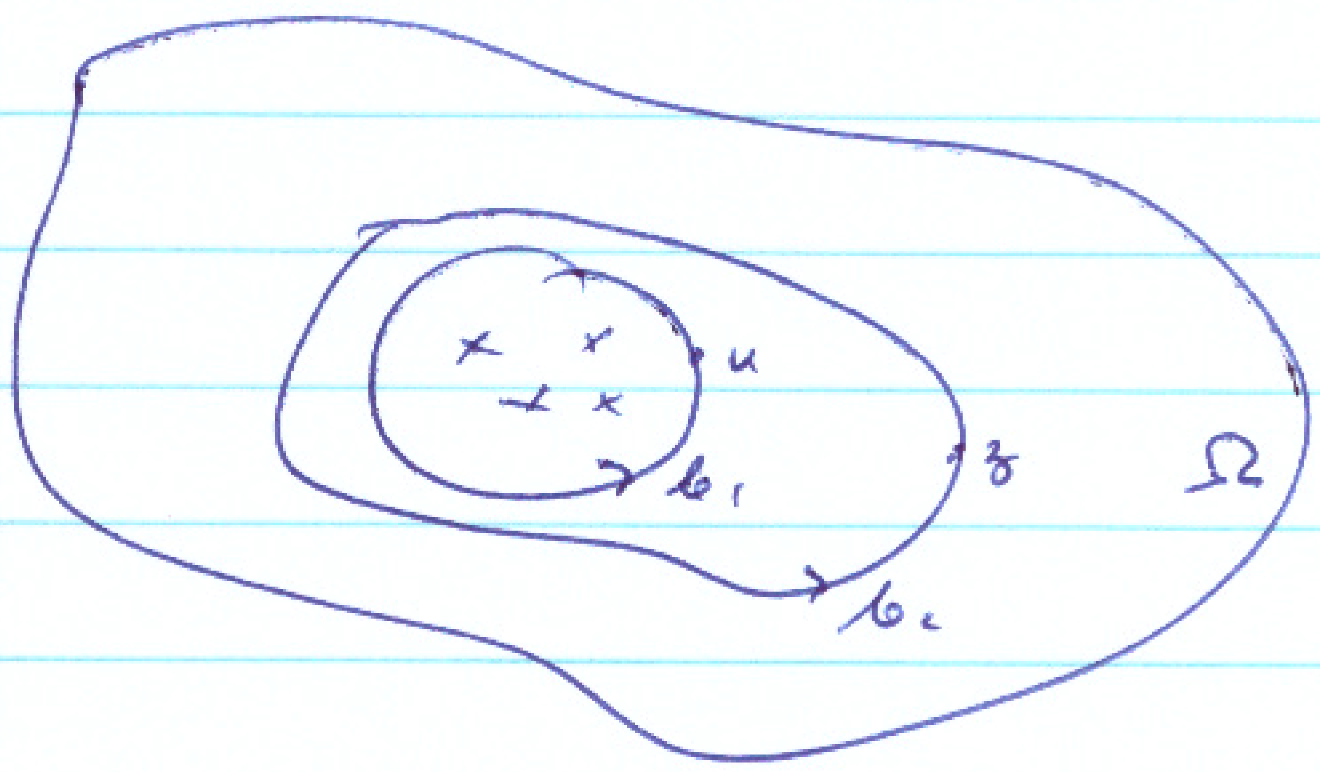
where $\frac{1}{z-A} = (z-A)^{-1}$ = resolvent of A at z. Note that $(z-A)^{-1}$ exists and is analytic on (a neighborhood) of $\mathcal{C}$, and hence is continuous on $\mathcal{C}$. Thus the contour integral in (66.1) exists.

As we are dealing with matrices, we can understand (66.1) coordinate-wise, i.e. $$(f(A))_{jk} = \frac{1}{2\pi i}\int_{\mathcal{C}} f(z)\left(\frac{1}{z-A}\right)_{jk} dz$$

However (66.1) is a good definition also in the case that A is a bounded operator in a Banach space, when spec (A) is no longer a finite set, but is in general a closed, compact set [c.f. the discussion at the end of Lecture 4 on analytic maps from $\mathbb{C}$ into a Banach space].

NB: By Cauchy's Theorem, $f(A)$ is independent of $\mathcal{C}$, as long as $\mathcal{C} \subset \Omega$ encloses the spectrum of A.

We now verify that $f \mapsto f(A)$ has properties (i)–(v). Properties (i) and (ii) are obvious. To prove (iii), choose 2 positively oriented contours $\mathcal{C}_1$ and $\mathcal{C}_2$ in Ω such that $\mathcal{C}_1$ contains $\sigma(A) = \text{spec } A$ in its interior and such that $\mathcal{C}_2$ contains $\mathcal{C}_1$ in its interior



We have for f, g analytic in Ω,

$$g(A)f(A) = \left(\frac{1}{2\pi i}\int_{\mathcal{C}_2} \frac{g(z)}{z-A}\,dz\right)\left(\frac{1}{2\pi i}\int_{\mathcal{C}_1} \frac{f(u)}{u-A}\,du\right)$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int_{\mathcal{C}_1}\int_{\mathcal{C}_2} dz\,du \;\; g(z) f(u) \frac{1}{z-A}\,\frac{1}{u-A}$$

$$= \frac{1}{(2\pi i)^2} \iint_{\mathscr{C}_1, \mathscr{C}_2} dz\, du\, \frac{g(z)\, f(u)}{u - z} \left(\frac{1}{z - A} - \frac{1}{u - A} \right)$$

$$= \frac{1}{(2\pi i)^2} \int_{\mathscr{C}_2} dz\, \frac{g(z)}{z - A} \int_{\mathscr{C}_1} du\, \frac{f(u)}{u - z}$$

$$- \frac{1}{(2\pi i)^2} \int_{\mathscr{C}_1} du\, \frac{f(u)}{u - A} \int_{\mathscr{C}_2} \frac{g(z)}{u - z}\, dz .$$

$$= \quad 0 \quad + \frac{1}{2\pi i} \int_{\mathscr{C}_1} \frac{f(u)\, g(u)}{u - A}\, du$$

$$= \quad (gf)(A)$$

which proves (iii)

On the other hand, for $f(z) = 1$

$$f(A) = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{1}{z - A}\, dz .$$

But as $(z - A)^{-1}$ is analytic outside of $\sigma(A)$, we can, by Cauchy's Theorem, evaluate $f(A)$ as follows:

$$f(A) = \lim_{R \to \infty} \frac{1}{2\pi i} \oint_{|z| = R} \frac{dz}{z - A} .$$

But for $|z| = R >> 1$.

$$\frac{1}{z - A} = \frac{1}{z} + \frac{1}{z(z - A)}$$

Now $\frac{1}{2\pi i} \oint_{|z| = R} \frac{dz}{z} = I$, and as $|z| >> R$

$$\left\| \frac{1}{z(z-A)} \right\| \leq \frac{C}{R^2} \qquad (\text{prove this})$$

Thus

$$\left\| \oint_{|z|=R} \frac{dz}{z(z-A)} \right\| \leq \oint_{|z|=R} \frac{C}{R^2} |dz|$$

$$= \frac{C}{R^2} \cdot 2\pi R$$

$$\to 0 \quad \text{as } R \to \infty$$

Thus

$$f(A) = I$$

This proves (iv).

Finally for $f(z) = z$

$$f(A) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{z}{z-A} \, dz$$

$$= \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{z - A + A}{z - A} \, dz$$

$$= \frac{1}{2\pi i} \left(\int_{\mathcal{C}} I + \int_{\mathcal{C}} \frac{A}{z-A} \, dz \right)$$

$$= 0 + A \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{dz}{z-A}$$

$$= A.$$

which proves (v).

The above functional calculus has another important property:

(vi) For f analytic in Ω, and $\sigma(A_n), \sigma(A) \subset \Omega$, if $A_n \to A$, then $f(A_n) \to f(A)$.

To prove this we will need a

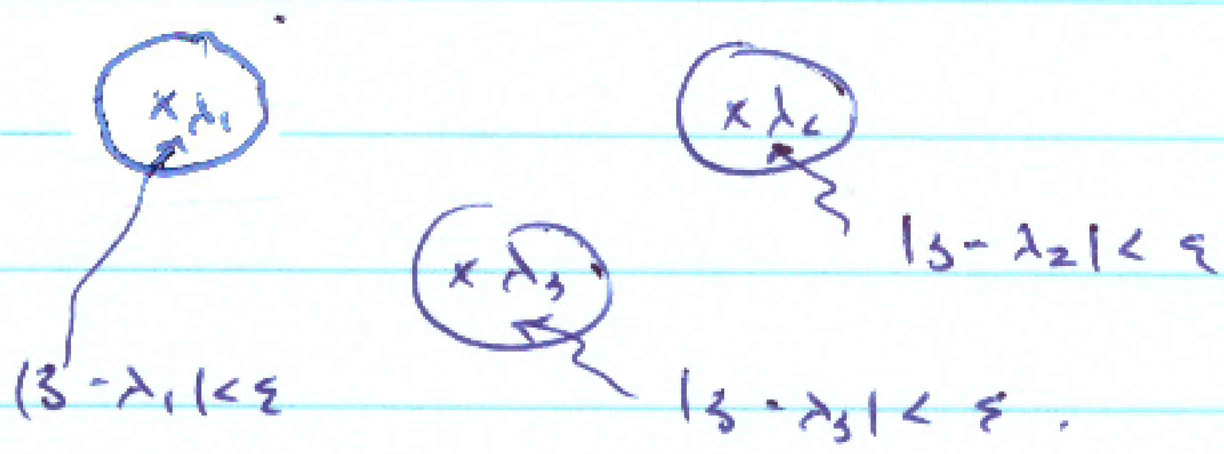
(70.1) Lemma

If $A_n \to A$, then $\sigma(A_n) \to \sigma(A)$ in the sense

that given any $\varepsilon > 0$, $\exists\, N$ st $n \geq N \Rightarrow$

$$\sigma(A_n) \subset \bigcup_{i=1}^{m} \{ z : |z - \lambda_i| < \varepsilon \}$$

where $\lambda_1, \dots, \lambda_m$ are the eigenvalues of A



Proof: For any $m \times m$ matrix B, $\det(z - B) = z^m + b_1 z^{m-1} + \dots + b_m$

is a polynomial and it is clear that $\exists\, R > 0$ st

$$|z| > R \Rightarrow |\det(z - B)| \geq$$

where R depends only on $\|B\|$. As $A_n \to A$, we see that for all n, $\|A_n\| \leq C$ for some C and hence $\exists\, R > 0$ st $|\det(z - A_n)| \geq 1$ for all $|z| \geq R$, for all n, where R is independent of n. On the other hand, given $\eta > 0$, it is clear that for $|z| \leq R$, $\exists\, N$ st

$n \geq N \Rightarrow \quad |\det(z - A_n) - \det(z - A)| < \eta/2$

Take $\quad \eta = \inf \{ |\det(z - A)| : |z| \leq R,\ |z - \lambda_i| \geq \varepsilon,\ i = 1, \dots, m \}$

As the roots of $\det(z - A)$ are the λ_i's, we must have $\eta > 0$.

It follows then that $|\det(z - A_n)| > \eta/2$ for $|z| \le R$ and $|z - \lambda_i| \ge \varepsilon$, $i = 1, \dots, m$.

But $|\det(z - A_n)| \ge 1$ for $|z| \ge R$. Thus the roots of $\det(z - A_n)$ must lie in the set

$$\bigcup_{i=1}^{m} \{|z - \lambda_i| < \varepsilon\}$$

whenever $n \ge N$. $\square$.

We now return to the proof of (vi). As $\sigma(A) \subset \Omega$, by lemma 70.1, $\sigma(A_n) \subset \Omega$ for sufficiently large n. Let $\mathcal{C} \subset \Omega$ be a fixed oriented contour with the property that $\sigma(A_n)$ and $\sigma(A)$ are contained in its interior for all sufficiently large n. We have

(71.1) $$f(A_n) - f(A) = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z) \left(\frac{1}{z - A_n} - \frac{1}{z - A} \right) dz$$

But for each $z \in \mathcal{C}$, $\frac{1}{z - A_n} \to \frac{1}{z - A}$ and the convergence is uniform in z. To see this we use the second resolvent identity for $z \in \mathcal{C}$

(71.2) $$\frac{1}{z - A_n} = \frac{1}{z - A} + \frac{1}{z - A_n} (A_n - A) \frac{1}{z - A}.$$

Now from Lecture 4, $\mathcal{C} \ni z \mapsto (z - A)^{-1}$ is continuous and hence $z \mapsto \|(z - A)^{-1}\|$ is continuous with $\sup_{z \in \mathcal{C}} \|(z - A)^{-1}\| = C < \infty$.

Thus

(72.0) $$\|(z-A_n)^{-1}\| \le \|(z-A)^{-1}\| + \|(z-A_n)^{-1}\| \, \|A_n - A\| \, \|(z-A)^{-1}\|$$

$$\le C\left(1 + \|(z-A_n)^{-1}\| \, \|A_n - A\|\right)$$

$$\le C + \tfrac{1}{2} \|(z-A_n)^{-1}\|$$

for n sufficiently large. Thus

(72.1) $$\|(z-A_n)^{-1}\| \le 2C$$

for all $z \in \mathcal{C}$ and n suff. large.

Thus from (71.1), using (71.2) and (72.0)

$$\|f(A_n) - f(A)\| \le \frac{1}{2\pi} \int_{\mathcal{C}} |f(z)| \; 2C^2 \|A_n - A\| \, |dz|$$

$$\le \frac{\text{length of } \mathcal{C}}{2\pi} \times \left(\max_{z \in \mathcal{C}} |f(z)|\right) \times 2C^2 \|A_n - A\|$$

$$\to 0 \text{ as } n \to \infty. \quad \square.$$

We now show that the functional calculus

$$(f, A) \to f(A)$$

with f analytic in Ω and $\sigma(A) \subset \Omega$ gives the "right" answer in certain cases.

(72.2) For example, suppose $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is entire.

So certainly f is analytic in $\mathcal{C}$. By (66.1)

$$f(A) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{z-A} \, dz = \frac{1}{2\pi i} \int_{\mathcal{C}} \left(\sum_{j=0}^{\infty} a_j z^j\right) (z-A)^{-1} \, dz$$

$$(73.1) \qquad = \sum_{j=0}^{\infty} a_j \frac{1}{2\pi i} \int_{\mathscr{C}} z^j (z-A)^{-1} dz$$

as $\sum_{j=0}^{\infty} a_j z^j$ converges uniformly.

Set $B \equiv \frac{1}{2\pi i} \int z (z-A)^{-1} dz$. Then by (iii)

$B^j = \frac{1}{2\pi i} \int z^j (z-A)^{-1} dz$. On the other hand,

by (v), $B = A$ and so $B^j = A^j$. It follows

then from (73.1) that $f(A) = \sum_{j=0}^{\infty} a_j A^j$ as desired

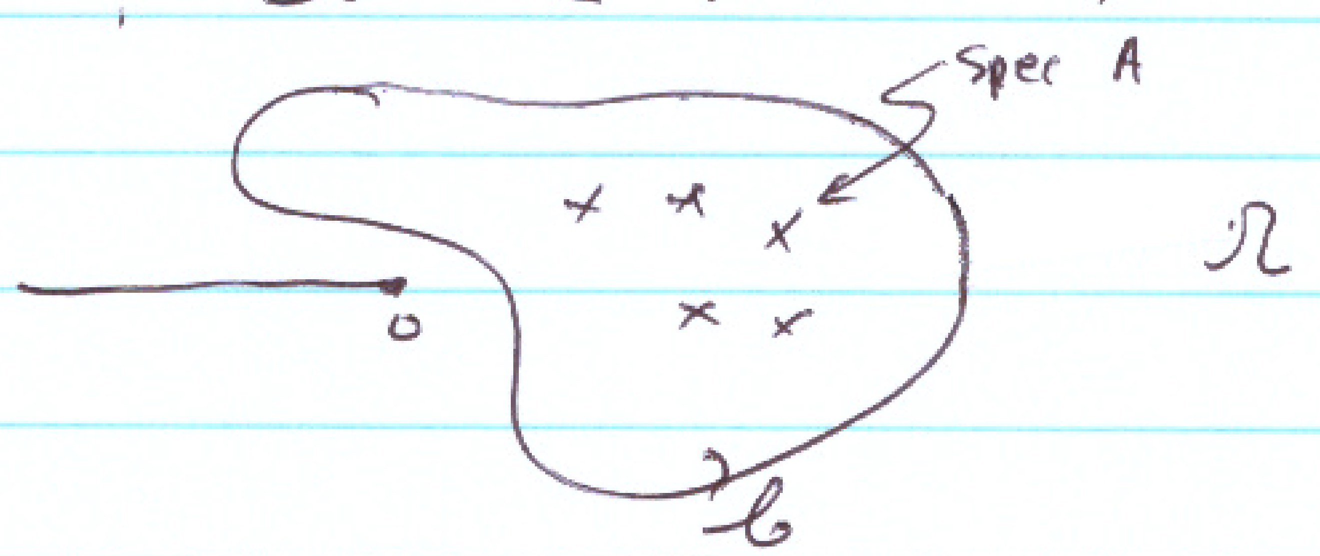
in (63.1) i.e. the new definition for $f(A)$ agrees with

the intuitive one when f is entire.

But we have gained a lot. We can speak

about $\log A$ and $\sqrt{A}$. More precisely, let

Ω be the slit plane, $\Omega = \mathbb{C} \setminus \{x < 0\}$



Then $\log z$ and $\sqrt{z}$ are certainly analytic in Ω. This means that we can construct $\log A$ and $\sqrt{A}$ for matrices with spectrum in the slit plane.

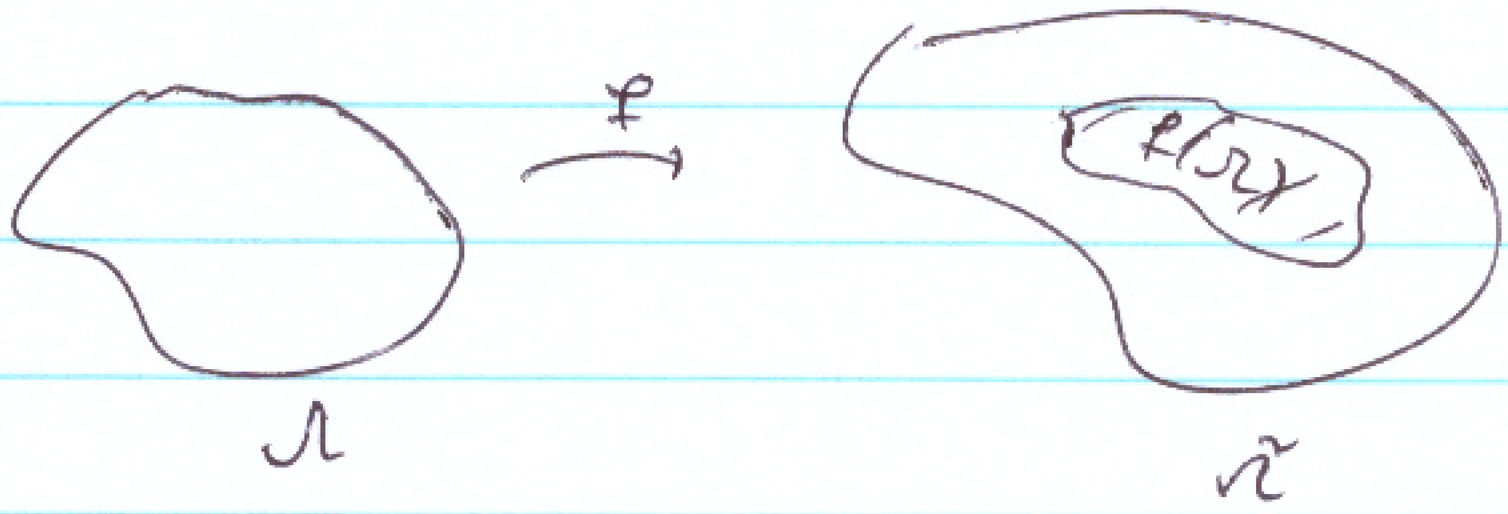
eg for A with $\sigma(A) \subset \mathbb{C} \setminus \{\text{slit}\}$,

$$\log A = \frac{1}{2\pi i} \int_{\mathcal{C}} \log z \, \frac{1}{z - A} \, dz .$$

where $\mathcal{C}$ is as above and $\log z$ is a branch of the logarithm analytic in Ω. Similarly for $\sqrt{A}$.

The functional calculus (f, A) also has the following property

(vii) If $f(z)$ is analytic in Ω, $g(\xi)$ analytic in $\tilde{\Omega}$ and $f(\Omega) \subset \tilde{\Omega}$



then

$$g(f(A)) = g \circ f\,(A) \tag{74.1}$$

for all A with $\sigma(A) \subset \Omega$