In order to prove (vii) we need the following result

Theorem 75.1 (spectral mapping theorem)

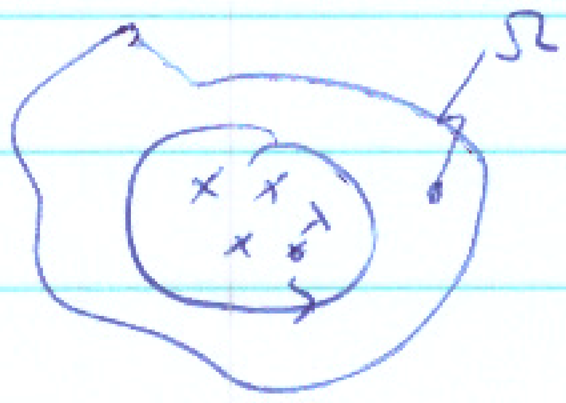
Suppose f is analytic in Ω and suppose $\sigma(A) \subset \Omega$.

Then $\sigma(f(A)) = f(\sigma(A)) = \{ z : z = f(\lambda) \text{ for } \lambda \in \sigma(A) \}$

Proof: Suppose $\lambda \in \sigma(A)$ and $Au = \lambda u$, $u \neq 0$.

Then for $z \in \rho(A)$, $(A - z)u = (\lambda - z) u$ and so

$(A - z)^{-1} u = \frac{1}{\lambda - z} u$. But then



$$f(A) u = \int_{\mathcal{C}} \frac{f(z)}{z - A} u \frac{dz}{2\pi i}$$

$$= \int_{\mathcal{C}} \frac{f(z)}{z - \lambda} u \frac{dz}{2\pi i} = f(\lambda) u .$$

Thus

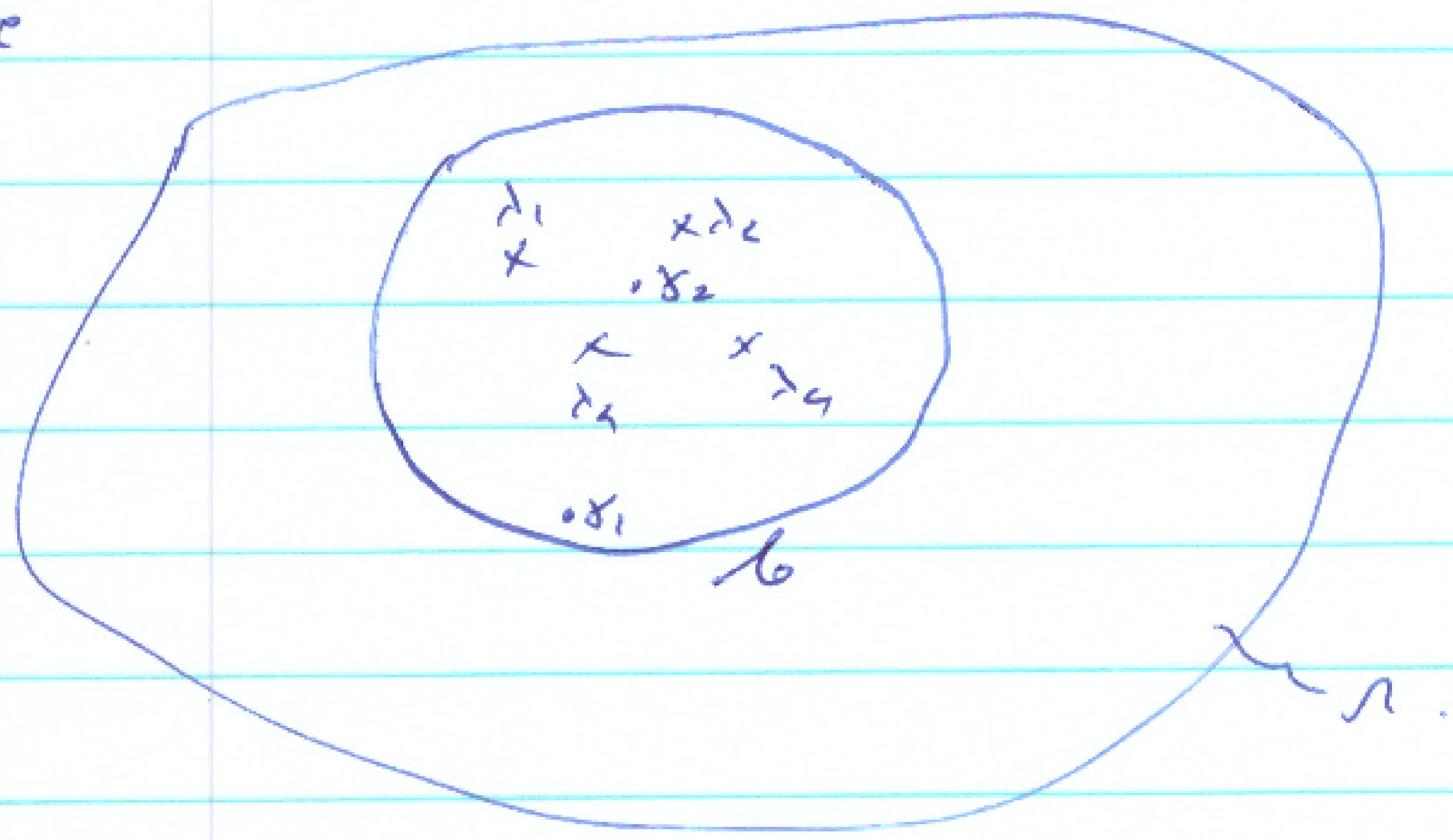
(75.2) $$f(\sigma(A)) \subset \sigma(f(A)) .$$

Now suppose $\mu \notin f(\sigma(A))$. We will show that $\mu \notin \sigma(f(A))$ and so $\sigma(f(A)) \subset f(\sigma(A))$. Hence by (25.2), we conclude that $f(\sigma(A)) = \sigma(f(A))$, as desired.

Let $\Gamma = \{\gamma \in \Omega : f(\gamma) = \mu\}$. By assumption $\Gamma \cap \sigma(A) = \emptyset$.

As f is analytic in Ω, it follows that $\overline{\text{int}(\Omega)} \cap \Gamma$ is finite

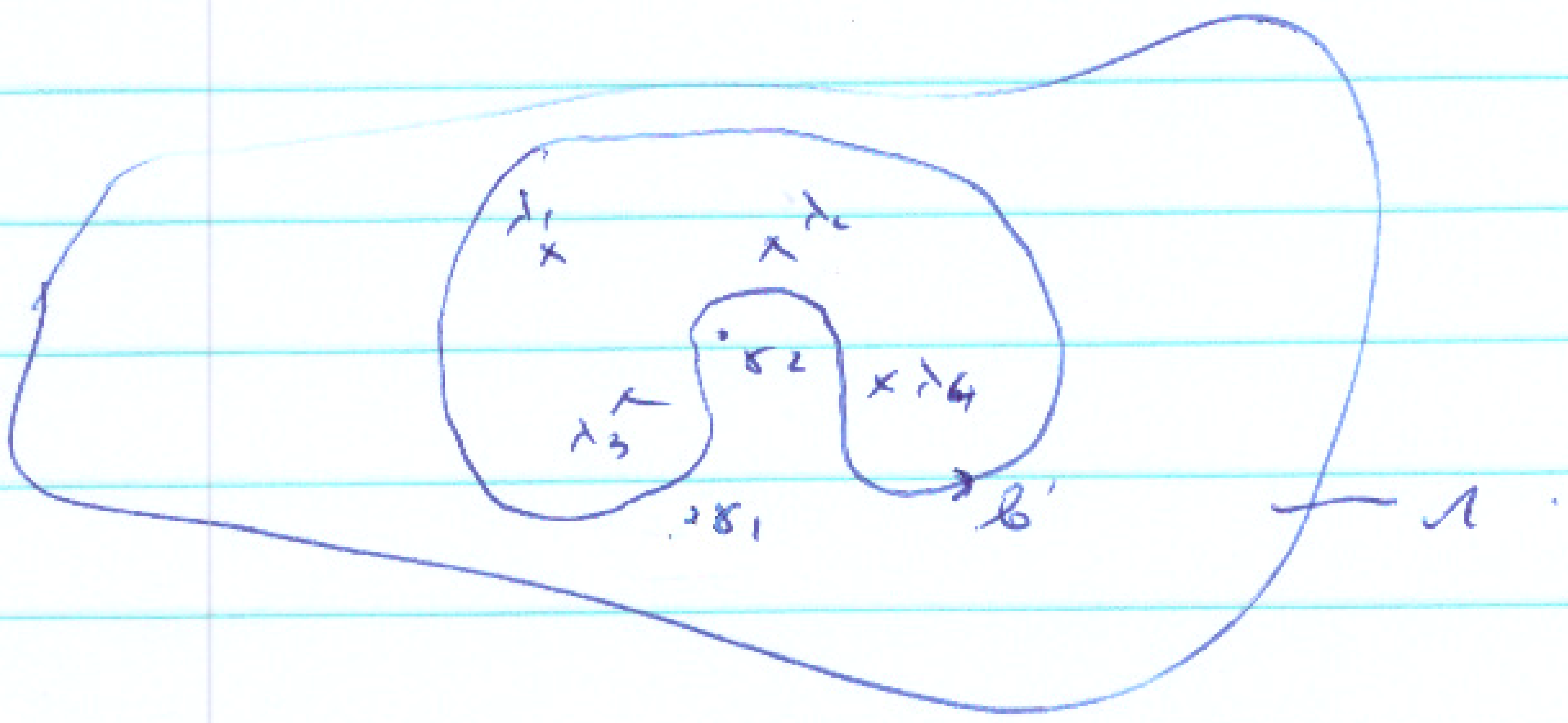
closure of $\text{int}(\Omega)$



Now it is easy to see that we can deform $\mathscr{C} \to \mathscr{C}'$

such that (exercise)

- $\sigma(A) \subset$ interior of $\mathscr{C}'$
- Γ lies in the exterior of $\mathscr{C}'$



It follows that

(77.1) $$f(z) - \mu \neq 0 \text{ in the interior of } \mathcal{C}' \text{ or on } \mathcal{C}'$$

Thus $(f(z)-\mu)^{-1}$ is analytic in interior of $\mathcal{C}'$ and on $\mathcal{C}'$

Set

$$G(A) = \frac{1}{2\pi i}\int_{\mathcal{C}'} \frac{1}{f(z)-\mu}\,\frac{1}{z-A}\,dz .$$

We have

$$f(A)-\mu = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(z)-\mu}{z-A}\,d\mu$$

$$= \frac{1}{2\pi i}\int_{\mathcal{C}'} \frac{f(z)-\mu)}{z-A}\,d\mu$$

and so by (iii)

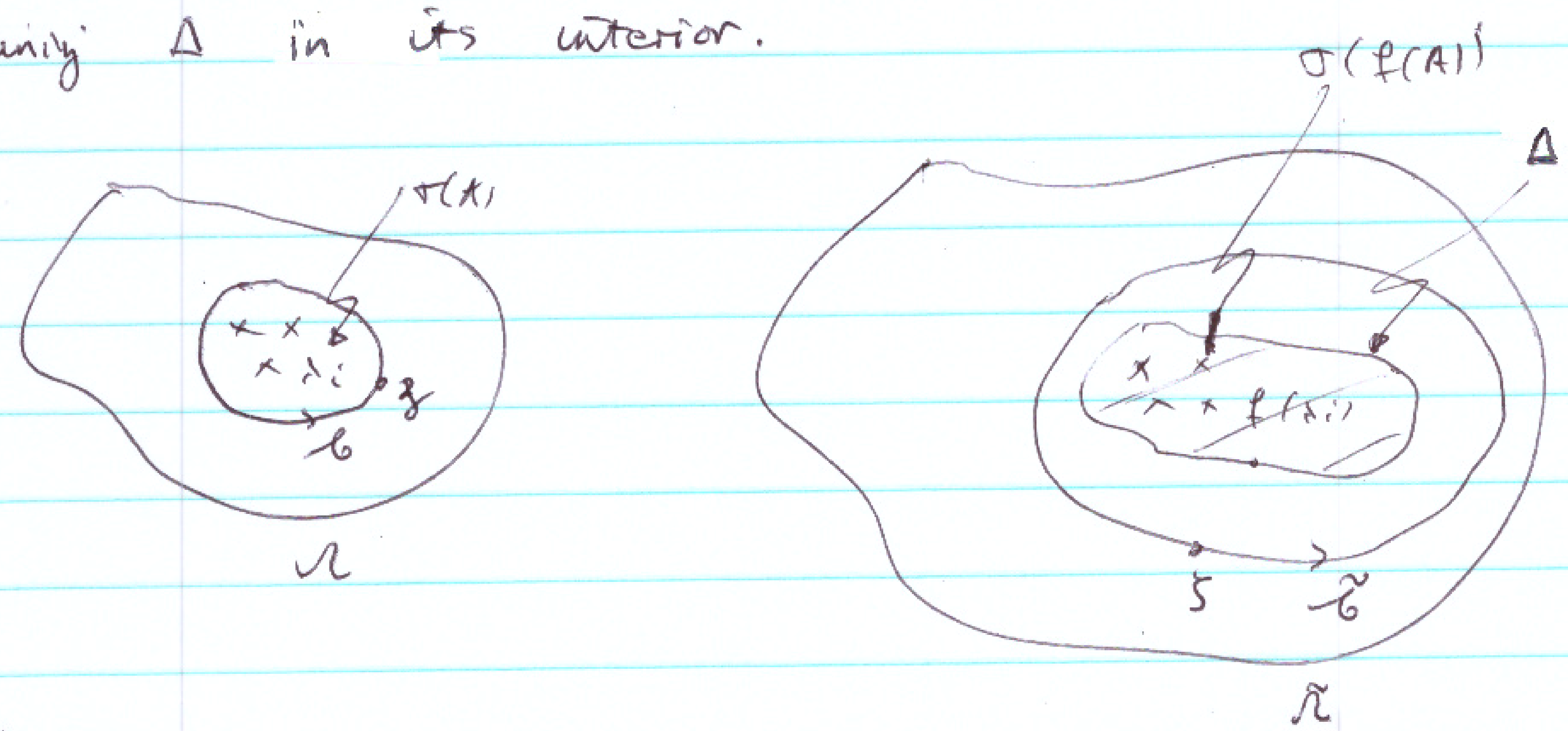
$$(f(A)-\mu)\,G(A) = (f-\mu)(f-\mu)^{-1}(A)$$
$$= 1(A)$$
$$= I$$

and similarly $G(A)\,f(A-\mu) = I$. Thus $\mu \notin \sigma(f(A))$ as desired. $\square$

We now prove (vii). We have

The proof we gave of (vii) in class only works, without further argument, only if f is 1-1 on Ω.

The following proof works directly even if f is not 1-1. Let $\mathcal{C} \subset \Omega$ be a contour such that $\sigma(A)$ is contained in $\operatorname{int} \mathcal{C}$. Let $\Delta = f(\overline{\operatorname{int} \mathcal{C}})$ where $\overline{\operatorname{int} \mathcal{C}}$ denotes the closure of $\operatorname{int} \mathcal{C}$. As f is certainly continuous on Ω, Δ is a compact set in $\tilde{\Omega}$ and let $\tilde{\mathcal{C}}$ be a contour in $\tilde{\Omega}$ containing Δ in its interior.



Now

(78.1) $$g(f(A)) = \frac{1}{2\pi i} \int_{\tilde{\mathcal{C}}} \frac{g(\xi)}{\xi - f(A)}\, d\xi$$

as $\sigma(f(A)) = f(\sigma(A)) \subset \Delta \subset \text{int}\,\tilde{\mathcal{C}}$, by Thm 75.1.

Now $z \mapsto \xi - f(z)$ is clearly an analytic,

non-zero analytic function in a neighborhood of $\overline{\text{int}\,\mathcal{C}}$

and hence $\dfrac{1}{\xi - f(z)}$ is analytic in this neighborhood, $\xi \in \tilde{\mathcal{C}}$

Thus

(79.1) $$\frac{1}{\xi - f(A)} = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{1}{\xi - f(z)}\,\frac{1}{z - A}\,dz .$$

Substituting (79.1) into (78.1) we find

$$g(f(A)) = \frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}} g(\xi)\left(\int_{\mathcal{C}} \frac{1}{\xi - f(z)}\,\frac{1}{z-A}\,\frac{dz}{2\pi i}\right)d\xi$$

$$= \int_{\mathcal{C}} \frac{1}{z-A}\left(\int_{\tilde{\mathcal{C}}} \frac{g(\xi)}{\xi - f(z)}\,\frac{d\xi}{2\pi i}\right)\frac{dz}{2\pi i}$$

$$= \int_{\mathcal{C}} \frac{1}{z-A}\, g(f(z))\,\frac{dz}{2\pi i},$$ by Cauchy's Theorem, as $f(z) \in \text{int}\,\tilde{\mathcal{C}}$,

$$= g \circ f(A),$$

as desired.

It follows, for example, from (vii) that

$$e^{\log A} = e^{\log}(A) = I$$

$$\log(e^{A}) = (\log\circ\exp)(A) = I$$

$$(\sqrt{A})^2 = (\sqrt{\ })^2(A) = A$$

The functional calculus gives a simple proof of

(80.1) Cayley-Hamilton Theorem. For a square matrix A, let

$$p(\lambda) = \det(\lambda - A)$$

be the characteristic polynomial of A. Then

$$p(A) = 0.$$

Proof: Choose Ω st $\sigma(A) \subset \Omega$. Then for $\mathcal{C}$ with $\sigma(A) \subset$ interior of $\mathcal{C}$,

$$p(A) = \int_{\mathcal{C}} \frac{p(z)}{z-A}\,\frac{dz}{2\pi i}$$

Now by Cayley's theorem $\left(\frac{1}{z-A}\right)_{ij} = \frac{\alpha_{ij}(z)}{p(z)}$

where $\alpha_{ij}(z)$ is the ij-cofactor of $z - A$. Thus

Thus

$$(P(A))_{ij} = \int_{\mathcal{C}} \frac{p(z)}{p(z)} \alpha_{ij}(z) \frac{dz}{2\pi i}$$

$$= \int_{\mathcal{C}} \alpha_{ij}(z) \frac{dz}{2\pi i}$$

$$= 0$$

as $\alpha_{ij}(z)$ is a polynomial in z and hence entire. $\square$.

<u>Example</u> Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then

$$p(z) = \det(z - A) = \det\begin{pmatrix} z-a & b \\ c & z-d \end{pmatrix}$$

$$= z^2 - (a+d)z + ad - bc.$$

We have

(81.1) $$P(A) = A^2 - (a+d)A + ad - bc$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - (a+d)A + (ad-bc)$$

$$= \begin{pmatrix} a^2+bc & b(a+d) \\ (a+d)c & bc+d^2 \end{pmatrix} - \begin{pmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{pmatrix}$$

$$+ \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= 0$$

The Cayley-Hamilton Theorem implies that if A is $n\times n$ then A^k can be expressed as a linear combination of $I, A, A^2, \ldots, A^{n-1}$. Indeed, by induction, if $A^k = a_0 + a_1 A + \cdots + a_{n-1} A^{n-1}$, then

(82.1) $$A^{k+1} = A(a_0 + a_1 A + \cdots + a_{n-1} A^{n-1}) = a_0 A + a_1 A^2 + \cdots + a_{n-2} A^{n-1} + A^n$$

But if $\det(\lambda - A) = \lambda^n + \gamma_{n-1}\lambda^{n-1} + \cdots + \gamma_0$, then by Cayley-Hamilton

(82.2) $$A^n + \gamma_{n-1} A^{n-1} + \cdots + \gamma_0 = 0$$

Thus from (82.1) (82.2)

$$A^{k+1} = a_0 A + \cdots + a_{n-1} A^{n-1} - \gamma_0 - \gamma_1 A - \cdots - \gamma_{n-1} A^{n-1}$$

$$= b_0 + b_1 A + \cdots + b_{n-1} A^{n-1}$$

for some $b_0, \ldots, b_{n-1}$

For example, from (81.1) for $a+d=0$

$$A^2 = \delta \quad , \text{ where } \quad \delta = bc - ad$$

Thus

$$A^{2k} = A^{2(k-1)} A^2 = A^{2(k-1)}\delta = \cdots = \delta^k$$

$$A^{2k+1} = A A^{2k} = \delta^k A$$

Thus

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

$$= \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!} A^{2j} + A \sum_{j=0} \frac{t^{2j+1}}{(2j+1)!} A^{2j}$$

$$= \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!} \delta^j + A \sum_{j=0}^{\infty} \frac{t^{2j+1}}{(2j+1)!} \delta^j$$

Now

$$\cosh t = \frac{e^t + e^{-t}}{2} = \frac{1}{2} \sum_{j=0}^{\infty} \frac{t^j}{j!} + \frac{1}{2} \sum_{j=0}^{\infty} \frac{(-1)^j t^j}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!}$$

and

$$\sinh t = \frac{d}{dt} \cosh t = \sum_{j=0}^{\infty} \frac{(2j) t^{2j-1}}{(2j)!}$$

$$= \sum_{j=1}^{\infty} \frac{t^{2j-1}}{(2j-1)!}$$

$$= \sum_{j=0}^{\infty} \frac{t^{2j+1}}{(2j+1)!}$$

Thus

(83.1) $$e^{tA} = \cosh t\sqrt{\delta}\, I + \frac{\sinh t\sqrt{\delta}}{\sqrt{\delta}} A$$

$$= \begin{pmatrix} \cosh t\sqrt{\delta} + a \sinh t\sqrt{\delta}/\sqrt{\delta} & b \sinh t\sqrt{\delta}/\sqrt{\delta} \\ c \sinh t\sqrt{\delta}/\sqrt{\delta} & \cosh t\sqrt{\delta} + d \sinh t\sqrt{\delta}/\sqrt{\delta} \end{pmatrix}$$

Note that if $\delta < 0$ $\cosh t\sqrt{\delta} \to \cos t\sqrt{|\delta|}$, $\sinh t\sqrt{\delta} \to i \sin t\sqrt{|\delta|}$

Thus the solution of

$$\frac{du}{dt} = Au, \qquad u(t=0) = u_0 \qquad (a+d=0)$$

has the form $u(t) = e^{tA} u_0$ as in (83.1). In particular

$u(t)$ is periodic if $\delta = bc - ad < 0$

Exercise Compute $u(t)$ in the case that $a+d \neq 0$

The functional calculus (66.1) defines $f(A)$

for functions f analytic in Ω and matrices A

with $\sigma(A) \subset \Omega$.

In the case that A is diagonalizable, and

in particular for normal matrices, the functional calculus

can be extended to all continuous functions $f: \mathbb{C} \to \mathbb{C}$.

More precisely, for any continuous function f and

any diagonalizable matrix A, set

(84.1) $$f(A) \equiv U \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix} U^{-1}$$

where U is the matrix of eigenvectors of U

are $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A.

Exercise

Check that (84.1) is a good definition i.e. $f(A)$ is independent of the order of the eigenvalues $\lambda_1, \dots, \lambda_n$. Also if two eigenvalues are equal, for example, say $\lambda_1 = \lambda_2$, then $f(A)$ is independent of the choice of independent eigenvectors corresponding to $\lambda_1 = \lambda_2$

Note that if f is analytic in Ω and $\sigma(A) \subset \Omega$, and A is diagonalizable, $A = U \Lambda U^{-1}$, $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, then from (66.1)

$$f(A) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(s)}{s-A}\, ds = \frac{1}{2\pi i} U \left(\int_{\mathcal{C}} \frac{f(s)}{s-\Lambda}\, ds \right) U^{-1}$$

$$= U \begin{pmatrix} \int_{\mathcal{C}} \frac{f(s)}{s-\lambda_1} \frac{ds}{2\pi i} & & 0 \\ & \ddots & \\ 0 & & \int_{\mathcal{C}} \frac{f(s)}{s-\lambda_n} \frac{ds}{2\pi i} \end{pmatrix} U^{-1}.$$

$$= U \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} U^{-1}, \text{ by Cauchy.}$$

which agrees with (84.1). Thus the functional calculus given by (84.1) extends the calculus given by (66.1).

Exercise 1

Prove properties (i) ... (vii) for the extended function calculus (84.1)

Exercise 2

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a + d = 0$,

from (81.1)

$\det(\lambda - A) = \lambda^2 - \delta$

Thus as long as $\delta \neq 0$, A has 2 distinct eigenvalues and hence A is diagonalizable.

Use (84.1) to compute $u(t) = e^{tA} u_0$ and compare with (83.1)

Jordan canonical form

We have shown that given any matrix A, $\exists\, U$, $\det U \neq 0$, and an upper triangular matrix S, st $A = U S U^{-1}$. What is the best we can do? We

We know that in general S cannot be diagonalized, i.e. in general, given S there may not be an invertible matrix U s.t. $S = U\Lambda U^{-1}$, Λ = diagonal matrix. We have instead,

<u>Jordan canonical form</u>

Given an $n \times n$ matrix A, $\exists$ an integer $k \le n$, and an invertible matrix U, and an upper triangular matrix J such that

(87.1) $$J = \begin{pmatrix} B_1 & 0 & & 0 \\ 0 & B_2 & & \\ & & \ddots & \\ 0 & & 0 & B_k \end{pmatrix}$$

where each B_i is an $n_i \times n_i$ matrix, $\sum_{i=1}^{k} n_i = n$ of the form

$$B_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$$

ie B_i has λ_i down the diagonal, and 1's on the first upper diagonal.

The λ_i's are the eigenvalues of A and, of course,

λ_i may equal λ_j for $i \neq j$. The <u>multiplicities</u> n_i are uniquely determined by A, apart from a reordering of the blocks B_i.

Conversely it is clear that if 2 matrices A and B have the same eigenvalues, and the same multiplicities n_i, then they are conjugate i.e. $\exists U$, $\det U \neq 0$ s.t. $A = UBU^{-1}$.

Thus Jordan form answers the natural question: When are 2 matrices conjugate? <u>Answer</u>: If and only if they have the same eigenvalues and the same multiplicities.

(<u>Note</u>: n_i is different in general from the algebraic and from the geometric multiplicity of an eigenvalue.)

Thus 2 matrices with the same eigenvalues, all distinct, are necessarily conjugate — a fact we could have proved directly (Why?) But $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, say are not conjugate, though their spectra are equal, because they have different multiplicities. Note also that

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are not conjugate, although they have the same eigenvalues.

We will outline a proof of the Jordan Form in an exercise.

Exercise Use Jordan form to describe the solution of $\frac{du}{dt} = Au$, $u(0) = u_0$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a + d = 0$, $\delta = bc - ad = 0$.

Compare your solution with (83.1) and Exercise 2 on p86.