Lecture 1

Riemann - Hilbert Problems

Fall 2009

References

(1) K. Clancey & I. Gohberg: Factorization of Matrix Functions and Singular Integral Operators Birkhauser, Oper. Th^y: Advances & Applications #3, 1981

(2) G. S. Litvinchuk & I. M. Spitkovskii: Factorization of Measurable matrix functions, Oper. Th^y: Adv. & Applic. #25, 1987

Special functions are important because they provide explicitly solvable models for a vast array of phenomena in mathematics. By "special functions" I mean Bessel functions, Airy functions, Legendre functions and so on.

It works like this. Consider the

Airy equation (see eg M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover Public's, New York, 1965)

(2.1) $$y''(x) = x\,y(x)$$

Seek a solution of (2.1) of the form

$$y(x) = \int_\Sigma e^{xs} f(s)\,ds$$

for some function $f(s)$ and some contour Σ in the complex plane. We have

$$y''(x) = \int_\Sigma s^2 e^{xs} f(s)\,ds$$

and

$$x\,y(x) = \int_\Sigma \left(\frac{d}{ds} e^{xs}\right) f(s)\,ds = -\int_\Sigma e^{xs} f'(s)\,ds,$$

provided we can drop the boundary term. In order to solve (2.1) we need to have

$$-f' = s^2 f$$

and so

$$f = \text{const}.\, e^{-\frac{1}{3}s^3}$$

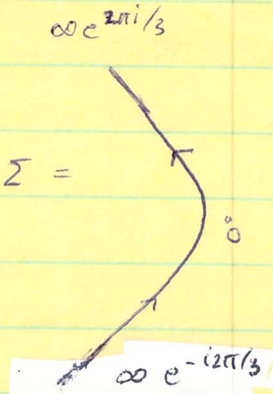
Thus

$$y(x) = \text{const} \int_\Sigma e^{xs - s^3/3}\,ds$$

provide solutions of the Airy equation.

The particular choice

$$\text{const} = \frac{1}{2\pi i}, \qquad \Sigma = $$



is known as Airy's integral $Ai(x)$

(3.1) $$Ai(x) = \frac{1}{2\pi i} \int_\Sigma e^{xz - \frac{1}{3}z^3}\, dz$$

Other contours provide other, independent solutions of (2.1), such as $Bi(x)$ (see Abram. & Stegun).
Now the basic fact of the matter is that

the integral representation (3.1) for $Ai(x)$ enables us,

using the classical method of stationary phase / steepest descent, to compute the asymptotics of

$Ai(x)$ as $x \to +\infty$ and $-\infty$ with any desired accuracy.

We find in particular (see Ab. + Steg p448) for $\boxed{\zeta \equiv \frac{2}{3} x^{3/2}}$

(3.2) $$Ai(x) \sim \frac{1}{2\sqrt{\pi}} x^{-\frac{1}{4}} e^{-\zeta} \sum_0^\infty (-1)^k c_k \zeta^{-k} \quad \text{as } x \to +\infty$$

$$c_0 = 1, \quad c_k = \Gamma(3k + \tfrac{1}{2}) \Big/ 54^k\, k!\, \Gamma(k + \tfrac{1}{2}) = \frac{(2k+1)(2k+3)\cdots(6k-1)}{(216)^k\, k!}$$

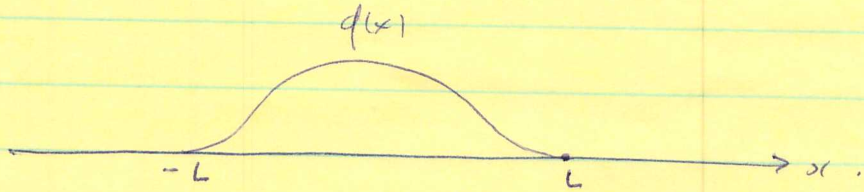
and

$$(4.0) \quad Ai(-x) = \frac{1}{\sqrt{\pi}} x^{-1/4} \left[\sin\left(\xi + \frac{\pi}{4}\right) \sum_0^\infty (-1)^k c_{2k} \xi^{-2k} - \cos\left(\xi + \frac{\pi}{4}\right) \sum_0^\infty (-1)^k c_{2k+1} \xi^{-2k-1} \right]$$

Such results for solutions of general second order equations are very rare. Consider for example the scattering problem for a one-dimensional Schrödinger operator

$$(4.1) \quad -y'' + q(x)\, y = \lambda^2 y \quad , \lambda > 0,$$

where the potential $q(x)$ has compact support.

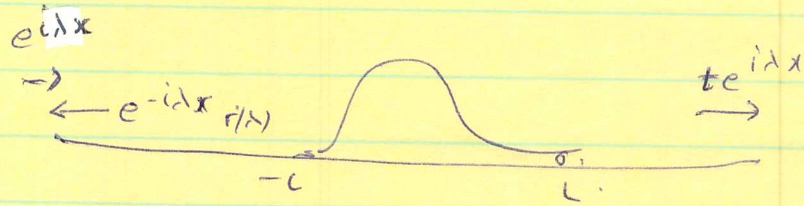


Consider the solution of (4.1) of the form $y(x) = e^{i\lambda x} c(\lambda)$ for some constant $c(\lambda)$, when $x > L$. Then $y(x)$ is of the form $a(\lambda) e^{i\lambda x} + b(\lambda) e^{-i\lambda x}$ for some $a(\lambda), b(\lambda)$

for $x < -L$. Rescaling

$$a(\lambda) \to 1$$
$$b(\lambda) \to b/a \equiv r$$
$$c(\lambda) \to c/a \equiv t.$$

This represents a wave $e^{i\lambda x}$ of unit size impinging on $q(x)$ from the left. ($f(x,t) = e^{i(\lambda x - \lambda^2 t)}$ solves the Schrodinger eqtn. $i\frac{\partial f}{\partial t} = Hf = -\frac{\partial^2 f}{\partial x^2}$)



An amount $t(\lambda)\, e^{i\lambda x}$ is transmitted and an amount $r(\lambda)\, e^{-i\lambda x}$ is reflected. It is clearly of great physical interest to be able to compute $t(\lambda)$, the transmission coefficient, and $r(\lambda)$ the reflection coefficient in terms of $q(x)$. However this is possible to do explicitly only in very special cases: in particular the formulae (3.2)(4.0) show that this is true for the Airy equation $q(x) = x$!

In the semi-classical limit for quantum mechanics

(6.1) $$-\hbar^2 y'' + q(x)\, y(x) = \lambda\, y(x), \quad \hbar \to 0$$

WKB analysis allows us to compute the eigenfunction of (6.1) with considerable accuracy as long as we are away from points x_0, say, such that $q(x_0) - \lambda = 0$: near x_0, $\hbar^2$ is no longer "small" compared to $q(x) - \lambda$. But in this region $q(x) - \lambda \sim \alpha x$ for some constant α, and so (6.1) reduces in this region to the Airy equation

$$-\hbar^2 y'' + \alpha x\, y(x) \simeq 0$$

Hence solutions of Airy are good models for Schrödinger equations near turning points, $q(x) - \lambda \sim 0$. In particular the precise formulae (3.2) (4.0) enable us to pass through the turning point with high accuracy as $\hbar \to 0$.

Now in recent years it has become clear that an extremely broad class of problems in mathematics, engineering and physics are described in various critical regimes by a new class of special functions, the so-called <u>Painlevé functions</u>. There are 6 <u>Painlevé equations</u> and we will say a lot more about them throughout the course. These equations are 2nd order but as opposed to the Airy, Bessel, ... equations which are linear, these equations are non-linear.

<u>Example 1</u>

Consider solutions of the modified Korteweg de Vries equation (MKdV)

(7.1) $$u_t - 6u^2 u_x + u_{xxx} = 0 \quad , \quad -\infty < x < \infty, \quad t \geq 0,$$

$$u(x, t=0) = u_0(x) \to 0 \quad \text{as } |x| \to \infty$$

Then as $t \to \infty$, $|x| \le c\, t^{1/3}$, $c < \infty$

(8.0) $$u(x,t) = \frac{1}{(3t)^{1/3}}\, p\left(\frac{x}{(3t)^{\frac{1}{3}}}\right) + O\left(\frac{1}{t^{2/3}}\right).$$

where $p(s)$ is a particular solution of the Painlevé II equation

(8.1) $$p''(x) = x\, p(x) + 2p^3(x)$$

(D+Zhou)

Example 2

Let $\pi \in S_N$ be a permutation of the numbers $1, 2, \ldots N$: $\pi = (\pi_1 \pi_2 \ldots \pi_N)$. We say that $\pi_{i_1}, \ldots, \pi_{i_k}$ is an increasing subsequence of π of length k if

$$i_1 < i_2 < \ldots < i_k$$

and

$$\pi_{i_1} < \pi_{i_2} < \ldots < \pi_{i_k}.$$

Thus if $N = 6$ and $\pi = (4\,1\,3\,2\,6\,5)$, then 125 and 136 are increasing subsequences of π of length 3.

Let $l_N(\pi)$ denote the length of a longest increasing subs. in π eg for $N=6$ and π as above, then $l_6(\pi) = 3$, which is the the length of the longest incr. subseq's 125 and 136. Equip S_N with uniform measure: Thus

$$\mathrm{Prob}(l_N \le n) = \frac{\#\{\pi : l_N(\pi) \le n\}}{N!}$$

<u>Question:</u> How does l_N behave statistically as $N, n \to \infty$?

<u>Theorem</u> ([Baik-Deift-Johansson: 1999]).

Centre and scale l_N as follows: $l_N \to \chi_N \equiv \frac{l_N - 2\sqrt{N}}{N^{1/6}}$

Then

$$\lim_{N\to\infty} \mathrm{Prob}(\chi_N \le s) = e^{-\int_s^\infty (x-s)\, u^2(x)\, dx}.$$

($\equiv$ Tracy - Widom distribution)

where $u(x)$ is the unique solution of Painlevé II (Hastings - McLeod solution!) normalized such that

$$u(x) \sim Ai(x) \quad \text{as} \quad x \to +\infty.$$

In a similar way, a whole slew of problems in math/phys/engineering are described by solutions of Painlevé equations. The **key question** now becomes:

Can we describe the solutions of the Painlevé equations as precisely as we can describe the solutions of the classical special functions such as Airy, Bessel, ...? In particular, can we describe the solutions asymptotically with arbitrary precision? In particular, can we solve the basic connection problem as in (3.2) (4.0):

known behavior as $x \to +\infty$, say

$$\Rightarrow$$

known behavior as $x \to -\infty$,

and vice versa?

At the technical level, connection formulae such as (3.2) (4.0) can be obtained because of the existence of an integral representation such as (3.1) for the solution. Once we have such a representation, the asymptotic behavior is obtained by applying the (classical) steepest descent method to the integral. We are led to the following questions:

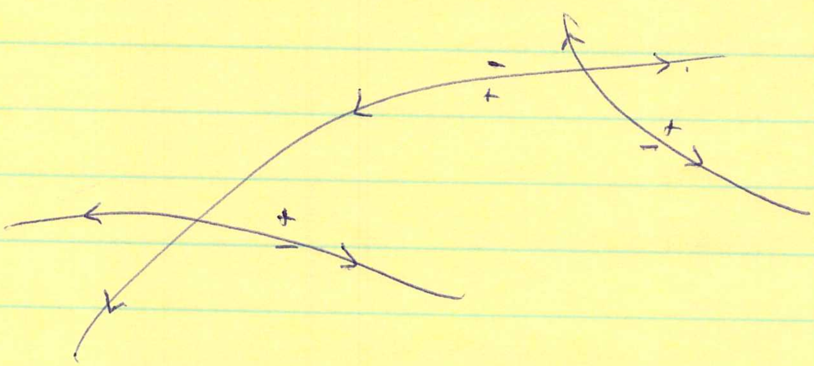
Q.1 Is there an analog of the integral representation for solutions of the Painlevé equations?

Q.2 Is there an analog of the steepest descent method which will enable us to extract precise asymptotic information about the Painlevé equations from this analog representation?

The answer to both questions is yes: in place of an integral representation, we have a

<u>Riemann-Hilbert Problem (RHP)</u>, and in place of the classical steepest descent method we have the <u>non-linear steepest descent method for RHP's</u> (introduced by D + Zhou, 1993).

So what is a RHP? Let Σ be an oriented curve in the plane



Σ

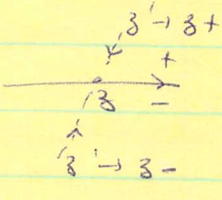
By convention, if we move along an arc in Σ in the direction of the orientation, the $(\pm)$-sides lie on the left (resp right). Let $v : \Sigma \to Gl(k, \mathbb{C})$ (the <u>jump matrix</u>) be an invertible $k \times k$ matrix function defined on Σ with $v, v^{-1} \in L^\infty(\Sigma)$.

We say that an $n \times k$ matrix function $m(z)$ is a solution of the RHP (Σ, v) if

- $m(z)$ is analytic in $\mathbb{C} \setminus \Sigma$

- $m_+(z) = m_-(z)\, v(z), \quad z \in \Sigma$

  where $m_\pm(z) = \lim_{z' \to z\pm} m(z')$



If, in addition, $n = k$ and

- $m(z) \to I_k$ as $z \to \infty$,

we say that $m(z)$ solves the <u>normalized</u> (RHP) (Σ, v)

RHP's involve a lot of technical issue which we will address in the coming lectures. In particular

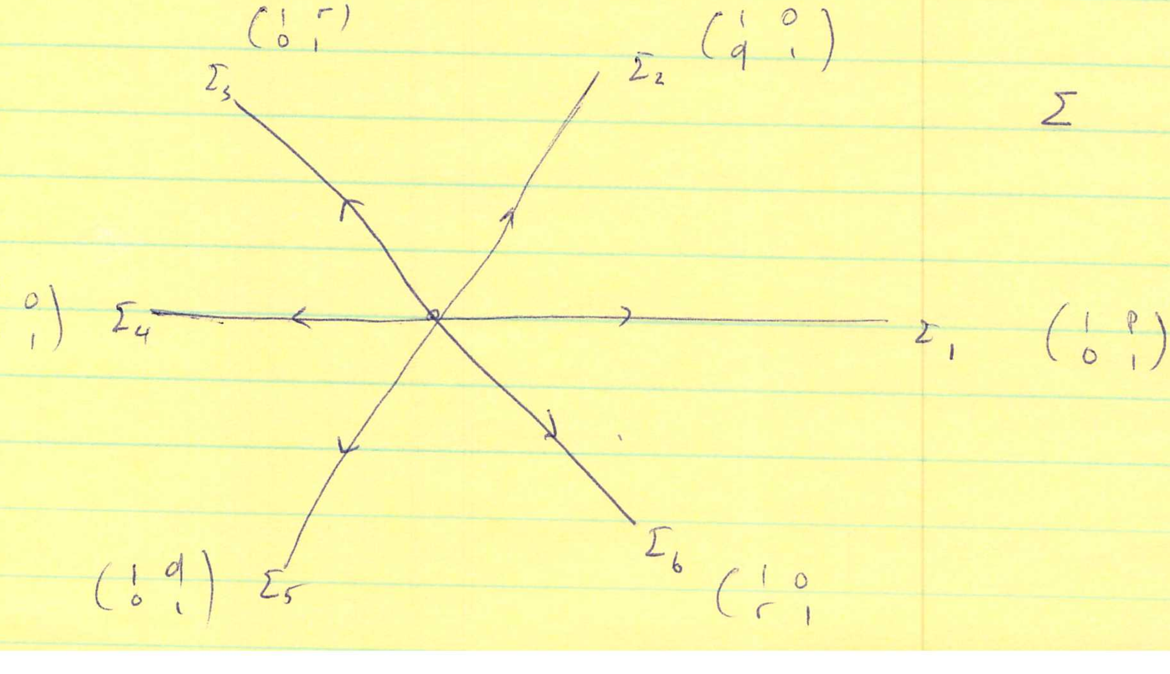
- how smooth should Σ be?
- in what sense are the limits $m_\pm(z)$ achieved?
- in what sense is the limit $m(z) \to I_k$ achieved, in the case $n = k$?
- does an $n \times k$ solution exist?
- in the normalized case, is the solution unique?
- at the analytical (i.e. equation) level, what kind of problem is a RHP? (As we will see, the

problem reduces to the analysis of singular integral equations on Σ.).

Leaving these matters aside for the moment, we now show how Painlevé II is related to a RHP. (see Painlevé Transcendents, the Riemann-Hilbert Approach.
A. S. Fokas, A. R. Its, A. A. Kapaev and V. Yu. Novokshenov,
Math. Surveys and Monographs Vol 128,
AMS, Providence Rhode Island, 2006.)

Let Σ denote the union of six rays $\Sigma_h = e^{i(h-1)\pi/3}$, $1 \le h \le 6$, oriented outward.

(Fig 14.1)



Let p, q, r be complex numbers satisfying the relation

$$p + q + r + pqr = 0$$

Let $v(z)$ be constant on each ray as indicated in Figure (14.1) and set for $z \in \Sigma$

$$v_x(z) = \begin{pmatrix} e^{-i(4z^3/3 + xz)} & 0 \\ 0 & e^{i(4z^3/3 + xz)} \end{pmatrix} v(z) \begin{pmatrix} e^{i(\frac{4z^3}{3} + xz)} & 0 \\ 0 & e^{-i(\frac{4z^3}{3} + x}\end{pmatrix}$$

Thus for $z \in \Sigma_3$.

$$v_x(z) = \begin{pmatrix} 1 & r e^{-2i(4z^3/3 + xz)} \\ 0 & 1 \end{pmatrix}$$

etc. Let $m_x(z)$ be the 2×2 matrix solution of the normalized RHP (Σ, v_x). Then.

(15.1) $$u(x) = 2i\,(m_1(x))_{12}$$

is a solution of Painlevé II where

(15.2) $$m_x(z) = I + \frac{m_1(x)}{z} + O\left(\frac{1}{z^2}\right) \text{ as } z \to \infty.$$

(Jimbo, Miwa and Ueno: Flaschka, Newell)

The asymptotic behavior of $u(x)$ as $x \to \infty$ is then obtained from the RHP (Σ, v_x) by the non-linear steepest descent methods.

One finds, e.g., (D + Zhou : Fokas et al).

Let $-1 < q < 1$, $p = -q$, $r = 0$. Then as $x \to -\infty$

(16.1) $$u(x) = \frac{\sqrt{2\nu}}{(-x)^{1/4}} \cos\left(\tfrac{2}{3}(-x)^{3/2} - \tfrac{3}{2}\nu \log(-x) + \phi\right) + O\left(\frac{\log(-x)}{(-x)^{5/4}}\right)$$

where

$$\nu = \nu(q) = -\frac{1}{2\pi} \log(1-q^2)$$

and

$\boxed{16.1'}$ $$\phi = -3\nu \log 2 + \log \Gamma(i\nu) + \frac{\pi}{2} \operatorname{sgn} q - \frac{\pi}{4}$$

(Γ = gamma function). As $x \to +\infty$,

(16.2) $$u(x) = q\,\mathrm{Ai}(x) + O\left(\frac{e^{-4/3\, x^{3/2}}}{x^{1/4}}\right)$$

Note the similarity of the multiplier $e^{i\left(\frac{4z^3}{3} + xz\right)}$ in the RHP (Σ, v_x) to the multiplier $e^{x z - \frac{1}{3} z^3}$ in (3.1). Clearly Painlevé II is the "natural" <u>non-linearization</u> of the Airy equation (cf (8.1) and (2.1)).

The RHP for the MKdV equation (7.1) is as follows (see D+Zhou): let $\Sigma = \mathbb{R}$ oriented from $-\infty$ to $+\infty$. For fixed $x, t \in \mathbb{R}$ let

$$v_{x,t}(z) = \begin{pmatrix} 1 - |r(z)|^2 & -\overline{r(z)}\, e^{-2i(4tz^3 + xz)} \\ r e^{2i(4tz^3 + xz)} & 1 \end{pmatrix}, \quad z \in \mathbb{R}$$

where r is a given function in $L^\infty(\mathbb{R})$ with $\|r\|_\infty < 1$, with $r(z) = -\overline{r(-z)}$, $z \in \mathbb{R}$.
Let $m_{x,t}$ be the solution of the normalized RHP $(\Sigma, v_{x,t})$. Then

$$(17.1) \qquad y(x,t) = +2\,(m_1(x,t))_{21}$$

where

$$m_{x,t}(z) = I + \frac{m_1(x,t)}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty.$$

The asymptotic result (8.0) is obtained by applying the non-linear steepest-descent method to the RHP $(\Sigma, v_{x,t})$ in the region $|x| \leq c\, t^{1/3}$. In this case Painlevé II emerges as the RHP $(\Sigma, v_{x,t})$ is "deformed" into the RHP (Σ, v_x) in Fig 14.1 for Painlevé II.

We will meet many more problems in math/phys/eng expressible in terms of a RHP.