Lecture 10

So far we have shown that for each x,

$Y(z) = Y(z; x, u(x), w(x) = u'(x))$ solves a RHP with

jump matrices $v_1 = v_1(x) = \begin{pmatrix} 1 & p(x) \\ 0 & 1 \end{pmatrix}$, $v_2 = v_2(x) = \begin{pmatrix} 1 & 0 \\ q(x) & 1 \end{pmatrix}$, ...

The question now is how do $p(x), q(x), r(x)$ move

in the case that $u(x)$, $w = u'(x)$ solve $P\text{II}$. We

know that

$$u(x) \text{ solves } P\text{II} \iff L_x = P_z + [P, L],$$
where $P = \begin{pmatrix} -iz & iu \\ -iu & iz \end{pmatrix}$

Now it follows from the general Th^m in Wasow that the solution

$Y(z)$ with standard asymptotics in a sector S

depends smoothly on the parameters $x, u(x), w = u'(x)$.

(Check this: exercise). In particular, $\frac{\partial Y}{\partial x}(z; x)$ $\exists$.

From $\frac{\partial Y}{\partial z} = LY$ we obtain

$$\frac{\partial}{\partial z}\frac{\partial Y}{\partial x} = \frac{\partial}{\partial x}\frac{\partial Y}{\partial z} = \frac{\partial}{\partial x} LY = L_x Y + L Y_x$$

$$= (P_z + PL - LP) Y + L Y_x$$

$$\therefore \quad \partial/\partial z (Y_x - PY) = (P_z + PL - LP) Y + L Y_x - P_z Y - P Y_z$$

$$= PLY - LPY - PLY + L Y_x$$

$$= L (Y_x - PY)$$

Hence

$$Y_x - PY = Y C$$

for some constant matrix C indep. of z. In any sector we have $Y = \hat{Y} e^{\theta\sigma_3} = (I + \frac{Y_1}{z} + \cdots) e^{\theta\sigma_3}$

as $z \to \infty$. Moreover the asymptotics can be differentiated wrt x (check!). Hence, as $\theta = -\left(\frac{4iz^3}{3} + izx\right)$,

$$\left(\hat{Y}(-iz\sigma_3) + \hat{Y}_x\right) e^{\theta\sigma_3} - \begin{pmatrix} -iz & iu \\ -iu & iz \end{pmatrix} \hat{Y} e^{\theta\sigma_3}$$

$$= \hat{Y} e^{\theta\sigma_3} C$$

$\Rightarrow$

$$iz[\sigma_3, \hat{Y}] - \begin{pmatrix} 0 & iu \\ -iu & 0 \end{pmatrix} \hat{Y} + O\left(\frac{1}{z}\right) = \hat{Y} e^{\theta\sigma_3} C e^{-\theta\sigma_3} \quad (143.1)$$

From (126.2) we have $Y_1^{12} = Y_1^{21} = u(x)/2$ and so

$$iz[\sigma_3, \hat{Y}] = i\begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} + O\left(\frac{1}{z}\right)$$

which $\Rightarrow$ LHS of (143.1) $= O\left(\frac{1}{z}\right)$ as $z \to \infty$ in the sector. But each of the 6 sectors contains a separation ray, and so we conclude that $C = 0$. (why?)

Hence

$$Y_x = PY$$

But differentiating $Y_+ = Y_- v$, we obtain

$$\frac{\partial Y_+}{\partial x} = \frac{\partial Y_-}{\partial x} v + Y_- \frac{\partial v}{\partial x}$$

$$\therefore \quad PY_+ = PY_- v + Y_- \frac{\partial v}{\partial x}$$

$$= PY_+ + Y_- \frac{\partial v}{\partial x}$$

and so

$$\frac{\partial v}{\partial x} = 0$$

We have thus proved the following basic result.

Thm 144.1 If u solves P.II, then the associated jump matrix $v(x, z)$ is indep of x i.e.

(144.2) $\qquad p(x) = \text{const}, \quad q(x) = \text{const}, \quad r(x) = \text{const}.$

The converse of Th^m 144.1 is true in the following form.

For $(p, q, r) \in V$, let v be the associated jump matrix on Σ ie $v(z) = v_1(z) = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$ for $z \in \Sigma_1$, etc. Set $v_x(z) = e^{\theta \sigma_3} v(z) e^{-\theta \sigma_3}$.

Th^m 145.1

Let $v = v(p, q, r)$ be as above. Suppose that for some $x_0 \in \mathbb{C}$ the normalized RHP (Σ, v_{x_0}) has a solution $m = m(z; x_0)$,

- $m_+(z; x_0) = m_-(z; x_0)\, v_{x_0}(z)$, $z \in \Sigma$
- $m_\pm - I \in \partial \mathcal{R}em\, \mathbb{C}(L^2)$, $m_\pm^{-1} - I \in \partial \mathcal{R}em\, \mathbb{C}(L^2)$.

Then there $\exists$ analytic functions $u = u(x)$, $w = u'(x) =$ defined in a neighborhood N_0 of x_0 such that

$$\frac{\partial Y}{\partial z} = L Y, \quad \frac{\partial Y}{\partial x} = P Y, \quad z \in \mathbb{C} \setminus \Sigma, \; x \in N_0 \tag{145.1}$$

where $Y = m e^{\theta \sigma_3} = \left(I + \frac{m_1(x)}{z} + \dots \right) e^{\theta \sigma_3}$

and

$$L = \begin{pmatrix} -4iz^2 - ix - 2iu^2(x) & 4u(x)iz - 2w(x) \\ -4u(x)iz - 2w(x) & 4iz^2 + ix + 2iu(x)^2 \end{pmatrix}$$

$$P = \begin{pmatrix} -iz & iu(x) \\ -iu(x) & iz \end{pmatrix}$$

In particular $u(x) = 2(m_1(x))_{12} = 2(m_1(x))_{12}$ solves $P\,II$

$$u'' = 2u^3 + xu$$

for $x \in N_0$. $\square$

The proof of this theorem is rather lengthy and we will not reproduce it here, but the key idea is the same old mantra we mentioned many times before viz $\Psi = m e^{\theta\sigma_3}$ solves a RHP on Σ

$\Psi_+ = \Psi_- v$ where v is <u>indep</u> of x and z.

Differentiation wrt x $\Rightarrow$ $\frac{\partial \Psi}{\partial x} = P\Psi$, and
differentiation wrt z $\Rightarrow$ $\frac{\partial \Psi}{\partial z} = L\Psi$, and finally
matching cross derivatives, $\partial_z \partial_x \Psi = \partial_x \partial_z \Psi$ $\Rightarrow$ $P\,II$.

The technicalities that need to be done are

(1) to prove that $\exists$ normalized solution of the RHP (Σ, v_x) for x in a nbhood N_0 of x_0

(2) to prove that the solution $Y = m e^{\theta\sigma_3}$ can be differentiated appropriately.

(3) to use Liouville to conclude $Y_x = PY$, $Y_3 = LY$ for the appropriate matrices P and L.

We now indicate how one proves the Painlevé property for $P\text{II}$.

Th^m 147.1 (Painlevé property)

Every solution $u(x)$ of $P\text{II}$,

(147-2)
$$u'' = 2u^3 + xu$$
$$u(x_0) = u_0, \quad u'(x_0) = u_0'$$

continues as a meromorphic function of x to all of $\mathbb{C}$.

In particular, any essential singularities, if any, of $u(x)$ are located at the point $x = \infty \in \overline{\mathbb{C}}$, indep. of u_0 or u_0'.

Again the proof is lengthy: we only indicate the key steps (see F.I.K.Nov. "Painlevé transcendents" for all the details).

Recall that a bded. operator A from a Banach space X to a Banach space Y is <u>Fredholm</u> if

(148.1) $\begin{cases} \text{(i)} & \ker A \text{ is finite dimensional} \\ \text{(ii)} & \operatorname{coker} A \text{ is finite dimensional} \end{cases}$

Here, $\ker A = \{u \in X : Au = 0\}$. Also

$\dim(\operatorname{coker} A) < \infty \iff \exists$ independent $y_1, \dots, y_n \in Y$, $n < \infty$, such that for any $y \in Y$, $\exists\, x \in X$ and $\alpha_1, \dots, \alpha_n$ in $\mathbb{C}$ such that $y = Ax + \sum_{i=1}^{n} \alpha_i y_i$. In this case $n = \dim(\operatorname{coker} A)$

The index of a Fredholm operator is defined by

(148.2) $$\operatorname{ind} A \equiv \dim(\ker A) - \dim(\operatorname{coker} A)$$

Key facts:

(148.3) If $A \in \mathcal{L}(X, Y)$ is Fredholm then $\exists\ \varepsilon > 0$ st

$A+B$ is Fredholm $\forall\ B \in \mathcal{L}(X,Y)$ with $\|B\| < \varepsilon$ and

$$\operatorname{ind}(A+B) = \operatorname{ind} A$$

(149.1) If $A \in \mathcal{L}(X,Y)$ is Fredholm, and $B \in \mathcal{L}(X,Y)$ is compact, then $A+B$ is Fredholm and

$$\operatorname{index}(A+B) = \operatorname{index} A$$

~~We will need the following result~~

~~Theorem (149.2) (Analytic Fredholm.~~

(149.2) <u>Pseudo-inverses</u>

$A \in \mathcal{L}(X,Y)$ is Fredholm if and only if $\exists$

$S_1, S_2 \in \mathcal{L}(Y,X)$

$$(149.3) \qquad S_1 A = \mathbb{1}_X + K_1 \quad , \quad A S_2 = \mathbb{1}_Y + K_2$$

for some <u>compact</u> operators $K_1 \in \mathcal{L}(X)$, $K_2 \in \mathcal{L}(Y)$.

<u>Example</u>: Suppose $K \in \mathcal{L}(X)$ is compact. Then $A = 1 + K$ is Fredholm. (Riesz–Schauder Theory!).

We will need the following basic and important result, which we present without proof:

Th^{m} 150.1 (Analytic Fredholm Th^{m}).

Suppose $z \to A(z)$ is an analytic map from an open connected set $D \subset \mathbb{C}$ into the Fredholm operators from a Banach space X to a Banach space Y.

Suppose

(150.2) $$A(z_0)^{-1} \ \exists \ \text{for some } z_0 \in D$$

Then $A(z)^{-1} \ \exists$ for all $z \in D \setminus J$ where J is a discrete subset of D and $A(z)^{-1}$ is meromorphic in D and analytic in $D \setminus J$ with finite rank residues at the points of J

Note: J discrete in D means that J has no accumulation pts in D.

The proof of Th^{m} 147.1 proceeds in the following way:

Step 1

~~Given x_0, $u_0 = u(x_0)$, $u_0' = u'(x_0)$, we construct~~

For any $(p, q, r) \in U$, construct

~~$p_0 = p(x_0, u_0, u_0')$, $q_0 = q(x_0, u_0, u_0')$, $r_0 = r(x_0, u_0, u_0')$~~

~~appearing in~~ the jump matrices $v_1 = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$, etc and normalized RHP

consider the $\Lambda(\Sigma, v_x)$ as above, where $v_x = e^{\theta\sigma_3} v e^{-\theta\sigma_3}$, $x \in \mathbb{C}$.

The solution $m_\pm = \psi_\pm e^{-\theta\sigma_3}$ of the RHP is computed

from the solution $\mu = \mu_x$, if it $\exists$, of the associated

singular integral equation on Σ,

(151.1) $$(1 - C_{\omega_x})\mu_x = I, \qquad \mu_x \in I + L^2(\Sigma),$$

where $C_{\omega_x} h = C^+ h\, \omega_x^- + C^- h\, \omega_x^+$ and

$v_x = (v_x^-)^{-1} v_x^+ = (I - \omega_x^-)^{-1}(I + \omega_x^+)$ is a pointwise

factorization of v_x. Indeed if μ_x solves (151.1)

then

(151.2) $$m_\pm = I + C^\pm[\mu_x(\omega_x^+ + \omega_x^-)] \in I + \partial C(L^2)$$

solves the normalized RHP (Σ, v_x). Step 1 consists

in showing that we can factorize $v = (v^-)^{-1} v^+$, and hence $v_x = (v^-)_x^{-1} (v^+)_x$, appropriately so that

$1 - C_{\omega_x}$ has a pseudo-inverse for all x , and

hence is Fredholm for all x

Step 2

In this step one shows that the Fredholm operator

$1 - C_{\omega_x}$ has index zero, $\mathrm{ind}\,(1 - C_{\omega_x}) = 0$ $\forall x \in \mathbb{C}$.

Step 3 Now let $x = x_0$, $u_0 = u(x_0)$, $u_0' = u'(x_0)$

as in (147.2) and consider the associated ~~jump~~

parameters

$$p_0 = p(x_0, u_0, u_0'), \; q_0 = q(x_0, u_0, u_0'), \; r_0 = r(x_0, u_0, u_0'),$$

$(p_0, q_0, r_0) \in V$, which comprise the jump matrix v^0,

$v_1^0 = \begin{pmatrix} 1 & p_0 \\ 0 & 1 \end{pmatrix}$, etc. The solution $\Psi(x_0, z)$ of $\frac{\partial \Psi}{\partial z} = L\Psi$

with standard asymptotics in $\mathbb{C} \setminus \Sigma$ gives rise to a

solution $m = m(x_0, z) = \Psi(x_0, z)\, e^{-\theta\sigma_3}$ of the normalized

RHP $(\Sigma, v_{x_0}^0)$, where $v_{x_0}^0 = e^{\theta_0\sigma_3} v^0 e^{-\theta_0\sigma_3}$, $\theta = \theta(x_0, z)$.

Taking $v = v^0$ in Step 3 we see that the operator $1 - C_{w^0_x}$ is Fredholm of index zero for all $x \in \mathbb{C}$. But for $x = x_0$ in particular, it follows from the existence of the solution $m(x_0, z)$ of the normalized RHP $(\Sigma, v^0_{x_0})$ that in fact $\ker(1 - C_{w^0_{x_0}}) = 0$. But $\dim \operatorname{coker}(1 - C_{w^0_{x_0}})$
$= -\operatorname{ind}(1 - C_{w^0_{x_0}}) + \dim \ker(1 - C_{w^0_{x_0}}) = 0 - 0 = 0$

Thus $1 - C_{w^0_{x_0}}$ is invertible.

Step 4 One sees easily that the map

$x \mapsto (1 - C_{w^0_x})$ is analytic in $\mathbb{C}$

Hence as $(1 - C_{w^0_{x_0}})^{-1}$ exists at $x = x_0$, it follows by the analytic Fredholm theorem that the map $x \to (1 - C_{w^0_x})^{-1}$ exists in $\mathbb{C} \setminus J$ for some discrete set $J \subset \mathbb{C}$ and, moreover, $(1 - C_{w^0_x})^{-1}$.

has finite dimensional residues at J.

Step 5

From the formula

$$m_+(x,z) = I + C^+ \mu(x,\cdot)(w_x^{0+} + w_x^{0-})$$

we see that as $m_+ = \Psi_+ e^{-\theta\sigma_3} = \hat{\Psi}_+(x,z)$

$$= I + \frac{\Psi_{1x}}{z} + \cdots,$$

$$\Psi_{1x} = -\int_\Sigma \mu(x,s)\,(w_x^{0+} + w_x^{0-})\,\frac{ds}{2\pi i}$$

$$= -\int_\Sigma \left(\frac{1}{1-C_{w_x^0}}(w_x^{0+} + w_x^{0-})\right)\frac{ds}{2\pi i}$$

$$= -\int_\Sigma (w_x^{0+} + w_x^{0-})\,\frac{ds}{2\pi i}$$

$$-\int_\Sigma \left[\frac{1}{1-C_{w_x^0}} C_{w_x^0}\right](w_x^{0+} + w_x^{0-})\,\frac{ds}{2\pi i}$$

from which it is clear that Ψ_{1x} is meromorphic in $\mathbb{C}$ with at worst poles of finite order at J, and hence the same is true for $u(x) = 2(\Psi_{1x})_{12}$. $\square$.

If $v = v_-^{-1} v_+ = (I - w_-)^{-1}(I + w_+)$ is any pointwise factorization of v and $C_w h = C^+ h w_- + C^- h w_+$ then we saw earlier that for $1 < p < \infty$,

$$1 - C_w \text{ is invertible in } L^p$$

$$\Longleftrightarrow$$

$IRHP1_{L^p}$ has a unique solution for each $f \in L^p$

$$m_+ = m_- v, \qquad m_\pm - f \in \partial C(L^p)$$

$$\text{and } \|m_\pm\|_{L^p} \le c \|f\|_{L^p}.$$

$$\Longleftrightarrow$$

$IRHP2_{L^p}$ has a unique solution for each $F \in L^p$

$$M_+ = M_- v + F, \qquad M_\pm \in \partial C(L^p)$$

$$\text{and } \|M_\pm\|_{L^p} \le c \|F\|$$

Recall that we say $m_\pm$ solves [illegible] the normalized RHP (Σ, v) if $m_\pm$ solves the $IRHP1_{L^p}$ with $f = I$ i.e.

$$m_+ = m_- v \text{ on } \Sigma \quad \text{and} \quad m_\pm - I \in \partial C(L^p).$$