Lecture 11

In our discussion of the Wiener-Hopf technique (lectures 6 & 7 : see also Remark on 101-102) the following issue arose: if we know that the normalized RHP (Σ, v) has a solution $m_\pm - I \in \partial C(L^p)$, how much do we know about $(1 - C_w)^{-1}$? If we write $m_\pm = I + h_\pm$, $h_\pm \in \partial C(L^p)$ then

$$h_+ = h_- v + F$$

where $F = v - I \in L^p(\Sigma)$. So if we know that the normalized RHP has a solution we know that the IRHP2$_{L^p}$ has a solution (only) for the special RHS $F = v - I$. How much more do we need to know about $m_\pm$ to conclude that $1 - C_w$ is a bijection? The following result addresses this question.

Thm 157.1 Let v, Σ be as above, with $v - I \in L^p(\Sigma)$. Let $C_v = C^- \circ (v - I)$ be the operator

corresponding to the (trivial) pointwise factorization

$v = (I)^{-1} v$, i.e. $v_+ = v$, $v_- = I$. Then

$1 - C_v$ is a bijection

$\Leftrightarrow$

the normalized RHP (Σ, v) has a
solution $m_\pm \in I + \partial C(L^p)$
such that $m_\pm^{-1} \in I + \partial C(L^q)$, $\frac{1}{p} + \frac{1}{q} = 1$
$1 < p, q < \infty$

and the map

$$T h \equiv (C^+ h (m_+)^{-1}) m_+$$

is bounded in $L^p(\Sigma)$.

Remark 157.2: Suppose Σ is bded. If $m(z)$ solves the normalized RHP (Σ, v) in the classical sense, and $\det m(z) \neq 0$, then $1 - C_v$ is invertible in $L^p(\Sigma)$ for all $1 < p < \infty$

Proof: Exercise.

Proof of Thm 157.1

Suppose $1 - C_v$ is a bijection. Then

the equation $(1 - C_v)\mu = I$ has a unique solution $\mu \in I + L^p(\Sigma)$. More precisely, if $\mu = I + h$, then $(1 - C_v)h = F$, $F = C_v I = C(v - I) \in L^p(\Sigma)$ has a unique solution with $h \in L^p(\Sigma)$. Standard computations then show that

$$m^\pm \equiv I + C^\pm(\mu(v-I))$$
$$= I + C^\pm(v - I) + C^\pm h(v - I)$$
$$\in I + \partial C(L^p)$$

solves the normalized RHP (Σ, v) in L^p.

Now for row vectors $h \in L^p$, the dual space consists of row vectors $g \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$, and the pairing of L^p and $L^q = (L^p)'$ may be realized by the inner product

(158.1) $$\langle h, g \rangle = \int_\Sigma h(s)\, g(s)^T\, ds$$

But as

$$C^\pm = \pm \frac{1}{2} + \frac{i}{2} H$$

where H is the Hilbert transform

$$Hh(z) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\substack{|z-s|>\varepsilon \\ s \in \Sigma}} \frac{h(s)}{z-s} \, ds \quad , \quad z \in \Sigma$$

we see that in the pairing (158.1)

$$\begin{aligned} (C^+)' &= +\tfrac{1}{2} + \tfrac{i}{2} H' \\ &= \tfrac{1}{2} - \tfrac{i}{2} H \\ &= -\left(-\tfrac{1}{2} + \tfrac{i}{2} H\right) \\ &= -C^- \end{aligned}$$

and similarly

$$(C^-)' = -C^+$$

Of course $(C^\pm)' \in \mathcal{L}(L^q)$

Now by general theory

$$1 - C_v \text{ is a bijection in } L^p$$

$$\Leftrightarrow$$

$$(1 - C_v)' \text{ is a bijection in } L^q .$$

We have $C_v = C^- R_{v-I}$ where R_{v-I}

denotes right multiplication by $v - I$, ie

$$R_{v-I} h \equiv h(v-I)$$

Also

$$\langle R_{v-I} h, g \rangle = \int h(v-I) g^T$$

$$= \int h \, (R_{v^T-I}\, g)^T$$

$$= \langle h, R_{v^T-I}\, g \rangle$$

so $R'_{v-I} = R_{v^T-I}$. Hence

$$(1 - C_v)' = (1 - C^- R_{v-I})' = 1 + R'_{v-I}\, C^+$$

$$= 1 + R_{v^T-I}\, C^+ .$$

But as noted before, by general theory

$1 + R_{v^T-I} C^+$ is a bijection

$\Leftrightarrow$

$1 + C^+ R_{v^T-I}$ is a bijection.

We have for $g \in L^q(\Sigma)$,

$$(1 + C^+ R_{v^T-I})g = g + C^+ g(v^T - I)$$

$$= g + C^- g(v^T-I) + g(v^T-I)$$

$$= (g v^T) - C^- g v^T (v^{-T} - I)$$

$$= (1 - C_{v^{-T}}) \circ R_{v^T} g$$

The above calculation imply that

$$1 - C_v \text{ is a bijection in } L^p(\Sigma)$$

$$\Leftrightarrow$$

$$1 - C_{v^{-T}} \text{ is a bijection in } L^q(\Sigma).$$

In particular we can conclude as above that the normalized RHP (Σ, v^{-T}) in L^q has a unique solution $\tilde{m}_\pm \in I + \partial C(L^q)$,

$$\tilde{m}_+ = \tilde{m}_- v^{-T}$$

But note that

$$m_+ \tilde{m}_+^T = m_- v \, v^{-1} \tilde{m}_-^T = m_- \tilde{m}_-^T$$

and by familiar arguments we see that $\tilde{m}_\pm^T = m_\pm^{-1}$ on Σ and so

$$m_\pm^{-1} \in I + \partial C(L^q).$$

Finally, for any $F \in L^p(\Sigma)$ the IRHPΣ_{L^p}

(161.1) $$m_+ = m_- v + F$$

has a unique solution $M_\pm \in \partial C(L^p)$ and

(162.2) $$\|M_\pm\|_{L^p} \le c\,\|F\|$$

(see p157). Setting $v = m_-^{-1} m_+$ in (161.1), we obtain

$$M_+ m_+^{-1} = M_- m_-^{-1} + F m_+^{-1}$$

and hence

$$M_\pm = \left(C^\pm F m_+^{-1}\right) m_\pm .$$

Note: $M_- = C^-(F v^{-1} m_-^{-1})\, m_-$ & so as $v, v^{-1} \in L^\infty$ $M_- = C^-(\tilde F m_-^{-1})\, m_-$ is bded in L^p

It follows then by (162.2) that T, in particular, is bded in $L^p(\Sigma)$, as desired.

Conversely, suppose that $m_\pm \in I + \partial C(L^p)$ solves the normalized RHP (Σ, v) with $m_\pm^{-1} \in I + \partial C(L^q)$ and $Th = \left(C^+\left(h (m^+)^{-1}\right)\right) m^+$ bded in L^p.

Consider the IRHP2$_{L^p}$

(162.3) $$\begin{cases} M_+ = M_- v + F, & F \in L^p \\ M_\pm \in \partial C(L^p) \end{cases}$$

Assume first that $F \in L^1 \cap L^\infty \subset L^p$ and set

$$\mathcal{M}_\pm = \left(C^\pm(F m_\pm^{-1})\right) m_\pm$$

Clearly

$$\mathcal{M}_+ = TF$$

and

$$\mathcal{M}_- = \left(C^-(F m_+^{-1})\right) m_+$$

$$= \left(C^+(F m_+^{-1}) - F m_+^{-1}\right) m_+$$

$$= TF - F$$

It follows that

(163.1) $$\|\mathcal{M}_\pm\|_{L^p} \le c\,\|F\|_{L^p}.$$

Let $h = F m_+^{-1}$. As $F \in L^1 \cap L^\infty$ and $m_\pm^{-1} \in I + \partial C(L^q)$,

it follows that $h \in L^1 \cap L^\infty + L^q \subset L^q$. But

then as $m_\pm \in I + \partial C(L^p)$, it follow that

$$\left(C^\pm(F m_+^{-1})\right) m_\pm = (C^\pm h)\, m_\pm = C^\pm k$$

when $k \in L^q + L^1$. But $k = C^+k - C^-k$

$= \mathcal{M}_+ - \mathcal{M}_-$ so that in fact $k \in L^p$. Hence

(163.2) $$\mathcal{M}_\pm \in \partial C(L^p).$$

Now

$$m_+ m_+^{-1} = C^+(F m_+^{-1}) = C^-(F m_+^{-1}) + F m_+^{-1}$$
$$= m_- m_-^{-1} + F m_+^{-1}$$

and so

$$m_+ = m_- m_-^{-1} m_+ + F = m_- v + F.$$

Thus for $F \in L^1 \cap L^\infty \subset L^p$ we have a unique (by Thm 68.3) solution of the IRHP2$_{L^p}$

$$\text{(164.1)} \quad \begin{cases} m_+ = m_- v + F \\ m_\pm \in \partial C(L^p), \quad \|m_\pm\|_{L^p} \leq c\,\|F\|_{L^p}. \end{cases}$$

In the proof we showed

$$m_\pm = C^\pm h = C^\pm(m_+ - m_-)$$
$$= C^\pm\left(TF - (TF - F)v^{-1}\right)$$

Thus given $F \in L^p$, choose $F_n \in L^1 \cap L^\infty$, $F_n \to F$ in L^p and ~~#~~ we conclude that (164.1) can be solved for any $F \in L^p$. In particular, $1 - C_v$ is a bijection in L^p. $\square$

Now what is the general relationship between $1-C_v$ and the normalized RHP (Σ, v) in the case that $1-C_v$ is not a bijection?

Consider the following simple example. Let $\Sigma = \{|z|=1\}$, and let $v = z^n$ on Σ,

Σ

$n \in \mathbb{Z}$.

Does a solution of the normalized RHP $\exists$,

(165.1)
$$m_+ = m_- v$$
$$m_\pm \in I + \partial C(L^2) \qquad ?$$

Suppose $n > 0$:

If $m_+ = m_- z^n$

Then we see that if $m(z)$ is the extension of $m_\pm$ off Σ, then

$$E(z) = m(z), \qquad |z| < 1$$
$$= m(z) z^n, \qquad |z| > 1$$

is entire. But as $m(z) = 1 + \int_\Sigma \frac{h(s)}{s-z} \frac{ds}{2\pi i}$, $h \in L^2$, we see that $E(z) = m(z) z^n = z^n + O(z^{n-1})$

as $z \to \infty$, and hence $E(z)$ is a monic polynomial $p(z)$

of degree n, $E(z) = z^n + \ldots \equiv p(z)$

Hence

$$
\begin{aligned}
m(z) &= p(z) \qquad |z| < 1 \\
&= \frac{p(z)}{z^n}, \qquad |z| > 1.
\end{aligned}
$$

Now if we require further that $m_\pm^{-1} \in I + \partial C(L^2)$,

then $m(z)$ cannot have any zeros in $\{|z| \le 1\}$
or in $\{|z| \ge 1\}$. But $p(z)$ has n zeros. This

is a contradiction. Hence (165.1) cannot have a

solution with $m_\pm$ and $m_\pm^{-1} \in I + \partial C(L^2)$ if $n > 0$.

Suppose $n = -\hat{n} < 0$. Then

$$
\begin{aligned}
E(z) &= m(z) \qquad |z| < 1 \\
&= m(z) z^n = \frac{m(z)}{z^{\hat{n}}}, \qquad |z| > 1.
\end{aligned}
$$

But then $E(z)$ is entire and as $n \to \infty$, $E(z) = O(\frac{1}{z^{\hat{n}}})$.
Thus $E(z) \equiv 0$ and so $m(z) \equiv 0$: in particular $m(z) \not\to 1$ as $z \to \infty$.

Thus in both cases a solution $m_\pm$ of the normalized RHP (Σ, v) does not $\exists$ for $n \neq 0$ with $m_\pm, m_\pm^{-1} \in I + \partial C(L^2)$.

Notice however that a solution of the following problem $\exists$:

(167.1) $$z^k m_+ = m_- v$$

where $k \in \mathbb{Z}$, and $m_\pm, m_\pm^{-1} \in I + \partial C(L^2)$.

Indeed take $k = n$ and $m_\pm = I$.

Now consider the IRHP2$_{L^2}$ corresponding to $v = z^n$ on $\Sigma = \{|z| = 1\}$.

(167.2) $$M_+ = M_- z^n + F, \qquad F \in L^2(\Sigma)$$
$$M_\pm \in \partial C(L^2).$$

Suppose $n > 0$. In this case, the solution of (167.2) cannot be unique. Indeed if

$$\hat{M}(z) = 1, \quad |z| < 1$$
$$= z^{-n}, \quad |z| > 1.$$

Then if $M_\pm$ solves (167.2), then so does $M_\pm + \hat{M}_\pm$.

How non-unique is $m_\pm$? This is clearly the dimension of the space $N = \{ m_\pm \in \partial C(L^2) : m_+ = m_- v = m_- z^n \}$

Now if

$$m_+ = m_- z^n \quad , \quad m_\pm = C^\pm h, \quad h \in L^2(\Sigma)$$

then

$$E(z) \equiv m(z) \quad , |z| < 1$$
$$\equiv m(z) z^n \quad , |z| > 1$$

is again a polynomial $p(z)$ by now of degree $n-1$.

Then

$$m(z) = p(z) \quad , \quad |z| < 1$$
$$m(z) = \frac{p(z)}{z^n} \quad , \quad |z| > 1$$

for any $p(z)$ of degree $\leq n-1$, gives a element of N.

Hence

$$\dim N = n$$

On the other hand given any $F \in L^2$, write

$$F = C^+F - C^-F = F_+ - F_-$$

We seek $m_\pm$ s.t. $m_+ = m_- z^n + F_+ - F_-$

Thus

$$M_+ - F_+ = M_- z^n - F_-$$

It follows then as before that

$$E(z) \equiv M(z) - CF(z) \quad , \quad |z| < 1$$

$$= M_- z^n - CF(z) \quad , \quad |z| > 1$$

$E(z)$ is entire and is of order $O(z^{n-1})$ as $n \to \infty$

Hence $E(z) = p(z)$ for some poly. $p(z)$ of degree $\leq n-1$. Thus

$$M(z) = CF(z) + p(z) \quad , \quad |z| < 1$$

$$= \frac{CF(z) + p(z)}{z^n} \quad , \quad |z| > 1$$

gives a solution of $M_+ = M_- v + F$, $M_\pm \in \partial C(L^2)$,

for any ~~poly~~ poly. $p(z)$ of deg $\leq n-1$.

In terms of the relation of the operator $1 - C_v$ and the IRHP2_{L^2}, the above calculations show (EXERCISE) that

$$\dim \ker (1 - C_v) = n$$

$$\dim \operatorname{coker} (1 - C_v) = 0$$

In particular $1 - C_v$ is Fredholm and

$$\operatorname{index} (1 - C_v) = n - 0 = n$$

Now consider (167.2) with $n < 0$; $\hat{n} = -n > 0$.

Suppose

$$\hat{m}_+ = \hat{m}_- z^n = \frac{\hat{m}_-}{z^{\hat{n}}} \quad , \hat{m}_\pm \in \partial C(L^2).$$

Then as before

$$E(z) = \hat{m}(z) \quad , |z| < 1$$
$$= \frac{\hat{m}(z)}{z^{\hat{n}}} \quad , |z| > 1$$

is entire and of order $O(z^{-(\hat{n}+1)})$ as $z \to \infty$. Hence

$\hat{m} \equiv 0$ $\therefore$ thus

(170.1). $$\dim N = 0$$

Now consider

$$m_+ = m_- z^{-\hat{n}} + F \quad , \quad F \in L^2 \quad , m_\pm \in \partial C(L^2)$$

Again set $F_\pm = C^\pm F$, $F_+ - F_- = F$

Thus we must consider

$$M_+ - C^+F = M_- z^{-\hat{n}} - C^- F$$

from which we see that

$$E(z) \equiv M(z) - CF(z) \quad , |z|<1$$
$$= \frac{M(z)}{z^{\hat{n}}} - CF(z) \quad , |z|>1$$

is entire. As $z \to \infty$, $E(z) \to 0$, Hence $E(z) \equiv 0$.
In particular

$$\frac{M(z)}{z^{\hat{n}}} = CF(z) \qquad \forall |z|>1 .$$

$$\text{ie} \quad M(z) = z^{\hat{n}} \, CF(z) = z^{\hat{n}} \int_\Sigma \frac{F(s)}{s-z} \frac{ds}{2\pi i}$$

$$= -z^{\hat{n}-1} \int_\Sigma F(s) \frac{1}{1-s/z} \frac{ds}{2\pi i}$$

$$= -z^{\hat{n}-1} \int_\Sigma F(s) \left(1 + \frac{s}{z} + \frac{s^2}{z^2} + \cdots + \frac{s^{\hat{n}-1}}{z^{\hat{n}-1}}\right) \frac{ds}{2\pi i}$$

$$+ O\left(\frac{1}{z}\right) \quad , \text{ as } z \to \infty$$

But $M(z) = O\left(\frac{1}{z}\right)$, so we must have

(175.1) $$\int_\Sigma F(s) s^j ds = 0 \quad , \quad 0 \le j \le \hat{n}-1 = |n|-1$$

In terms of the relation of the operator $1 - C_v$ to the IRHP2_{L^2}, the above calc's now show (exercise)

$$\dim \ker (1 - C_v) = 0$$

$$\dim \operatorname{coker} (1 - C_v) = |n| = -n.$$

In particular $1 - C_v$ is again Fredholm and

$$\operatorname{index} (1 - C_v) = 0 - |n| = n$$

Of course in the case $n = 0$, we have again

$$\dim \ker (1 - C_v) = 0$$
$$\dim \operatorname{coker} (1 - C_v) = 0$$

and $\operatorname{ind} (1 - C_v) = 0 = n$.

~~Exercise~~

~~Consider the following matrix example on~~ $\cancel{\Sigma = \{|z| = 1\}}$,

$$\cancel{v(z) = \begin{pmatrix} z^{-1} & 1 \\ 0 & z \end{pmatrix}}$$

~~(i) Verify that~~ $\cancel{v(z)}$ ~~has a factorization of the form~~