Lecture 9

$$\hat{Y}(z) = \hat{Y}_0 + \frac{\hat{Y}_1}{z} + \frac{\hat{Y}_2}{z^2} + \cdots \tag{121.1}$$

where $\det \hat{Y}_0 \neq 0$

Corollary (121.2)

If $A(z)$ is holo. at ∞, then a soln. Y of the form in the above Thm $\exists$ in every sector S with central angle less than $\pi/q+1$.

Notation: We say that any solution of (118.3) satisfying (120.2) (121.1) has standard asymptotics in S.

Some examples

(i) Suppose $Y' = A_0 Y$, 2×2 ; here $q = 0$ & $A(z) = A_0$

where A_0 has distinct eigenvalue $\lambda_1 \neq \lambda_2$, $\neq 0$

$A_0 = U \Lambda U^{-1}$, $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2)$, $\det U \neq 0$.

Then $Y(z) = U e^{z\Lambda} U^{-1}$

Hence $\tilde{Y}(z) = U e^{z\Lambda}$ is a fundamental solution of (i)

Clearly

$$\tilde{Y} = \hat{Y}(z)\, z^{D}\, e^{Q(z)}$$

when $D = 0$, $\hat{Y}(z) \equiv I = Y_0$, $\det I \neq 0$.

$$Q(z) = \frac{z^{0+1}}{0+1}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} = z\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

Here we are not restricted to an angle $\frac{\pi}{q+1} = \pi$.

(ii) Suppose

$$Y' = A_0 Y$$

as in (i) but now $A_0 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$

Here $q = 0$

Then $$Y = \begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}$$

is a fundamental solution of (i). Suppose $Y' = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} Y$ had a fundamental solution of form (120.2) in a sector S: then

$$Y = \hat{Y}(z)\begin{pmatrix}z^{d_1} & 0\\ 0 & z^{d_2}\end{pmatrix} e^{Q(z)} A = \begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}$$

for some matrix A, $\det A \neq 0$, and d_1, d_2 constants.

Then $Q = z \operatorname{diag}(0,0) = 0$, and so $\hat{Y} = \begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix} A^{-1}\begin{pmatrix}z^{-d_1} & 0\\ 0 & z^{-d_2}\end{pmatrix}$

(wlog "Q_0" $= 0$);

But this (exercise) is clearly incompatible with $\hat{Y} = \hat{Y}_0 + \frac{\hat{Y}_1}{z} + \dots$, $\det \hat{Y}_0 \neq 0$. So the rep. (120-2) can break down if the eig's of A_0 are not distinct (see Wasow for more information).

(iii) Consider 2×2 exple

$$\frac{dY}{dz} = z^q A_0 Y \qquad , \quad q \in \mathbb{N}_0, \quad A(z) = A_0$$

when A_0 has distinct eig's $\lambda_1 \neq \lambda_2$,

$$A_0 = U \,\mathrm{diag}(\lambda_1, \lambda_2)\, U^{-1}, \qquad \det U \neq 0 .$$

Then

$$Y(z) = U e^{\frac{z^{q+1}}{q+1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}} = U e^{Q}$$

is a fundamental solution. Indeed

$$Y' = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} z^q e^{Q}$$

$$= A_0 U z^q e^{Q}$$

$$= z^q A_0 Y ,$$

(iv) Consider $Y' = \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} Y \Rightarrow Y = \begin{pmatrix} e^{z} & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} e^{z} & 0 \\ 0 & 1 \end{pmatrix}$

We now apply the Th^m to (118.3), $\Psi' = L\Psi$;

$$L = z^2\left(A_2 + \frac{A_1}{z} + \frac{A_0}{z^2}\right), \quad q = 2,$$

in any sector S of opening angle $< \frac{\pi}{q+1} = \frac{\pi}{2+1} = \frac{\pi}{3}$

In such a sector we have a solution

(124.1) $$\Psi = Y = \hat{Y} z^D e^{Q(z)}$$

where

(124.2) $$\begin{cases} Q(z) = Q_3 z^3 + Q_2 z^2 + Q_1 z, \quad Q_i \text{ diagonal} \\ \qquad (\text{w.l.o. generality } "Q_0" = 0) \\ D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \\ \hat{Y} = Y_0 + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \dots, \text{ as } z \to \infty \text{ in } S \\ \det Y_0 \neq 0 \end{cases}$$

Moreover the asymptotics ~~for~~ $\hat{Y}$ can be differentiated and so inserting (124.1) into $Y' = LY$ we obtain

$$\frac{dY}{dz} = \left[\left(-\frac{Y_1}{z^2} - 2\frac{Y_2}{z^3} \dots\right) + \left(Y_0 + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \dots\right)\left(Q_1 + 2Q_2 z + 3Q_3 z^2 + \frac{D}{z}\right)\right] \times z^D e^{Q(z)}$$

$$= \left(A_2 z^2 + A_1 z + A_0\right)\left(Y_0 + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \dots\right) z^D e^{Q(z)}$$

Thus

$$3Y_0 Q_3 z^2 + (3Y_1 Q_3 + 2Y_0 Q_2) z + (3Y_2 Q_3 + 2Y_1 Q_2 + Y_0 Q_1)$$

$$+ (3Y_3 Q_3 + 2Y_2 Q_2 + Y_1 Q_1 + Y_0 I) z^{-1} + \dots$$

$$= A_2 Y_0 z^2 + (A_2 Y_1 + A_1 Y_0) z + (A_2 Y_2 + A_1 Y_1 + A_0 Y_0)$$

$$+ (A_2 Y_3 + A_1 Y_2 + A_0 Y_1) z^{-1} + \dots$$

<u>Order z^2</u> $\quad 3Y_0 Q_3 = A_2 Y_0$

But $\quad Q_3 = \frac{1}{3}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

where λ_1, λ_2 are the eig's of $A_2 = -4i\sigma_3$

(wlog, we can always assume that the eig's of Q_3 are in the same order as the eigs of A_2. why?)

$$\therefore \lambda_1 = -4i \quad , \lambda_2 = 4i$$

$$\therefore \quad Q_3 = \frac{-4i}{3}\sigma_3 = \frac{1}{3} A_2$$

Thus we have $\quad [Y_0, A_2] = 0$

and so Y_0 is diagonal. Hence as diag. matrices commute,

we can assume wlog that

(125.1) $$Y_0 = I$$

Order z $\quad 3Y_1 Q_3 + 2Y_0 Q_2 = A_2 Y_1 + A_1 Y_0$

$$\therefore \quad -4i\,[Y_1, \sigma_3] = -2Q_2 - 4u\sigma_2$$

$$\therefore \quad -4i \begin{pmatrix} 0 & -2Y_1^{12} \\ 2Y_1^{21} & 0 \end{pmatrix} = -2Q_2 - 4u\sigma_2$$

Hence as Q_2 is diag, and σ_2 is off-diag, $\Rightarrow$

(126.1) $$Q_2 = 0$$

and

$$-4i \begin{pmatrix} 0 & -2Y_1^{12} \\ 2Y_1^{21} & 0 \end{pmatrix} = -4u \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$\Rightarrow$

(126.2) $$u = 2Y_1^{12} = 2Y_1^{21}$$

Order z^0 $\quad 3Y_2 Q_3 + 2Y_1 \underbrace{\cancel{Q_2}}_{=0} + \underbrace{Y_0}_{=1} Q_1 = A_2 Y_2 + A_1 Y_1 + A_0 \underbrace{Y_0}_{=I}$

(126.3) $\Rightarrow \quad -4i\,[Y_2, \sigma_3] = -Q_1 + A_1 Y_1 + A_0$

$$= -Q_1 - 4u\,\sigma_2 Y_1 - (ix + 2iu^2)\sigma_3 - 2w\,\sigma_1$$

Now

$$\sigma_2 Y_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} Y_1 = \begin{pmatrix} -iY_1^{21} & -iY_1^{22} \\ iY_1^{11} & iY_1^{12} \end{pmatrix},$$ we see that

$$Q_1 = \operatorname{diag} Q_1 = -4u \begin{pmatrix} -iY_1^{21} & 0 \\ 0 & iY_1^{12} \end{pmatrix} - (ix + 2iu^2)\,\sigma_3$$

$$= \begin{pmatrix} 2u^2 i - ix - 2iu^2 & 0 \\ 0 & -2u^2 i + ix + 2iu^2 \end{pmatrix} = -ix\sigma_3$$

The ~~off-diagonal~~ diagonal elements of (126.3) $\Rightarrow$

$$-4i \begin{pmatrix} 0 & -2Y_2^{12} \\ 2Y_2^{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4ui\, Y_1^{22} \\ -4ui Y_1^{11} & 0 \end{pmatrix} + (-2w) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

or

(127.1) $$\begin{cases} -4i\; Y_2^{12} = -2iu\; Y_1^{22} + w \\ 4i\; Y_2^{21} = 2iu\; Y_1^{11} + w \end{cases}$$

Finally, <u>order z^{-1}</u>

$$-4i\,[Y_3, \sigma_3] + Y_1 Q_1 + D = A_1 Y_2 + A_0 Y_1$$

$$\Rightarrow \quad D = \operatorname{diag}\left(A_1 Y_2 + A_0 Y_1 - Y_1 Q_1 \right)$$

$$= \operatorname{diag}\Big(-4u\sigma_2 Y_2 + \left[-(ix + 2iu^2)\sigma_3 - 2w\sigma_1 \right] Y_1 + ix\, Y_1 \sigma_3 \Big)$$

$$= \left(\begin{pmatrix} 4ui\, Y_2^{21} & 0 \\ 0 & -4ui\, Y_2^{12} \end{pmatrix} + \begin{pmatrix} -2iu^2\, Y_1^{11} & 0 \\ 0 & 2iu^2 Y_1^{22} \end{pmatrix} \begin{pmatrix} -2w\, Y_1^{21} & 0 \\ 0 & -2w\, Y_1^{12} \end{pmatrix} \right)$$

$= 0$ by (126.2) and (127.1)

Assembling these results we see that as $z \to \infty$ in S

(128.1) $$Y(z) = \vec{Y}(z)\, e^{-[4iz^3/3 + ixz]\sigma_3} = \left(Y_0 + \frac{Y_1}{z} + \dots\right) e^{-\left(\frac{4iz^3}{3} + ixz\right)\sigma_3}$$

where S has opening angle $< \frac{\pi}{3}$. We also can take $Y_0 = I$.

Observe from

$$Y = \left(\vec{Y}_1 \;\; \vec{Y}_2\right) \begin{pmatrix} e^{-(i4z^3/3 + ixz)} & 0 \\ 0 & e^{(i4z^3/3 + ixz)} \end{pmatrix}$$

$$= \left(\vec{Y}_1\, e^{-(i4z^3/3 + ixz)} \quad \vec{Y}_2\, e^{(i4z^3/3 + ixz)}\right)$$

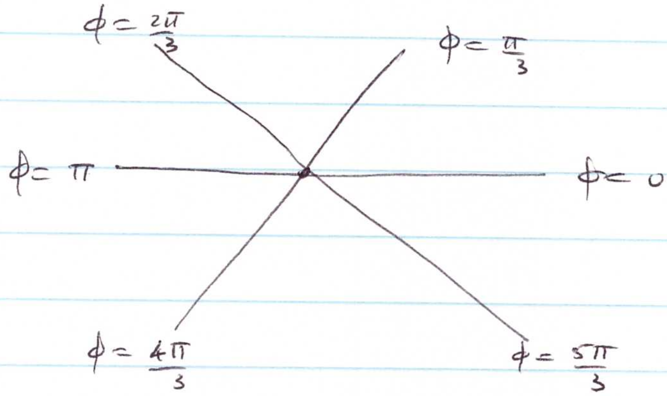
that the relative growth rates of the columns of Y as $z \to \infty$ are governed by $\mathrm{Re}\left[-(4iz^3/3 + ixz)\right]$ and $\mathrm{Re}\left[\frac{4iz^3}{3} + ixz\right]$ resp. What is important is where these rates become equal (to leading order) i.e.

(128.1) $$\mathrm{Re}\left[\left(-\frac{i4z^3}{3}\right) - \left(\frac{i4z^3}{3}\right)\right] = 0$$

Setting $z = |z| e^{i\phi}$ this condition becomes

(128.2) $$\sin 3\phi = 0$$

Thus the 6 rays



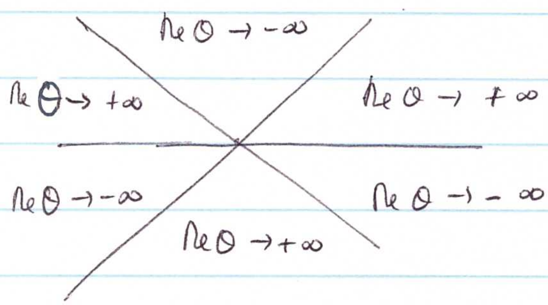
are important. These rays are called separation rays for our equation (118.3), $\frac{\partial \psi}{\partial z} = (z^2 A_2 + z A_1 + A_0)\psi$. (see Wasow p 84)

Set

(129.1) $$\Theta = -\left(\frac{4iz^3}{3} + ixz\right)$$

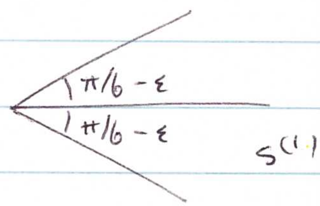
As $z \to \infty$ in the 6 sectors we see

(129.2)



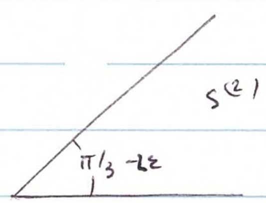
We now show (see Wasow p84) that if we choose the sectors judiciously to avoid separation rays, then the validity of the asymptotics in (128.3) can be extended to regions of opening angle $< 2\pi/3$.

We proceed as follows. Suppose $Y^{(1)}$ is a solution of (118.3) with standard asymptotics in the sector $S^{(1)}$



$Y^{(1)} = \hat{Y}^{(1)} e^{\theta\sigma_3}$, any $\varepsilon > 0$,

and let $Y^{(2)}$ be a solution with standard asymptotics in the sector $S^{(2)}$,



$Y^{(2)} = \hat{Y}^{(2)} e^{\theta\sigma_3}$, any $\varepsilon > 0$.

Now we must have

(130.1) $$Y^{(1)} = Y^{(2)} C \quad \text{ie} \quad \hat{Y}^{(1)} e^{\theta\sigma_3} = \hat{Y}^{(2)} e^{\theta\sigma_3} C.$$

for some constant matrix C. Hence

$$(\hat{Y}^{(2)})^{-1} \hat{Y}^{(1)} = e^{\theta\sigma_3} C e^{-\theta\sigma_3} = \begin{pmatrix} c_{11} & c_{12} e^{2\theta} \\ c_{21} e^{-2\theta} & c_{22} \end{pmatrix}$$

Letting $z \to \infty$ along the ray $\arg z = \frac{\pi}{12}$, say, which lies in $S^{(1)} \cap S^{(2)}$, we see that necessarily

$$\lim_{\substack{z \to \infty \\ \arg z = \frac{\pi}{12}}} \begin{pmatrix} c_{11} & c_{12}\, e^{2\theta} \\ c_{21}\, e^{-2\theta} & c_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From (129.2), we see that $c_{11} = c_{22} = 1$, $c_{12} = 0$

Thus

(131.0) $$Y^{(1)} = Y^{(2)} \begin{pmatrix} 1 & 0 \\ c_{21} & 1 \end{pmatrix}$$

$$\Rightarrow \quad \hat{Y}^{(1)} = \hat{Y}^{(2)} \begin{pmatrix} 1 & 0 \\ c_{21}\, e^{-2\theta} & 1 \end{pmatrix}$$

and as $\hat{Y}^{(2)} = I + \frac{\hat{Y}_1^{(2)}}{z} + \cdots$ in $0 \le \arg z \le \frac{\pi}{3} - \varepsilon$

we see that in fact

(131.1) $Y^{(1)}$ has standard asymptotics in

$$-\frac{\pi}{6} + \varepsilon < \arg z < \frac{\pi}{3} - \varepsilon.$$

A similar argument in the lower half-plane $\Rightarrow$ in fact

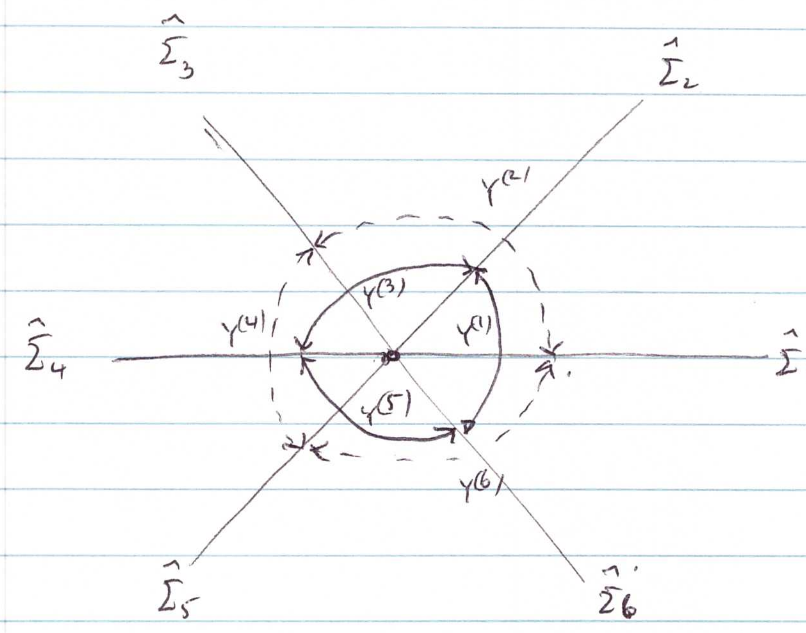
(131.2) $\left\{ \begin{array}{l} Y^{(1)}(z) \text{ has standard asymptotics in the sector} \\ -\frac{\pi}{3} + \varepsilon < \arg z < \frac{\pi}{3} - \varepsilon \quad \text{, any } \varepsilon > 0. \end{array} \right.$

This argument can be repeated at each of the 6 separation rays. Hence we can construct 6 solutions of the diff. eqtn. which we denote

$$Y^{(1)}, Y^{(2)}, \ldots, Y^{(6)}$$

which have standard asymptotics in sectors of opening angle $\frac{2\pi}{3} - \varepsilon$, $\varepsilon > 0$, centered around the rays $\hat{\Sigma}_1, \ldots, \hat{\Sigma}_6$ whr

$$\hat{\Sigma}_k = e^{(k-1)\pi/3}\,\mathbb{R}_+, \qquad k = 1, \ldots, 6.$$



Note that each of these solutions is <u>uniquely determined</u>. Indeed, any solution with standard asymptotics in a sector containing a separation ray is uniquely determined.

For example, if $\tilde{Y}$ and $Y^{\#}$ were 2 solutions in a sector S containing the separation ray $\hat{\Sigma}_1$, then

$$\tilde{Y} = Y^{\#} \tilde{C}$$

for some constant matrix $\tilde{C}$, then letting $z \to \infty$ along a ray $\arg z = \text{const} > 0$ in S, we see that $\tilde{C} = \begin{pmatrix} 1 & 0 \\ \tilde{c}_{21} & 1 \end{pmatrix}$ as in (131.0). And letting $z \to \infty$ along a ray $\arg z = \text{const} < 0$ in S, we see that $\tilde{c}_{21} = 0$, and so $\tilde{Y} = Y^{\#}$. By our mantra, the uniquely determined solutions $Y^{(j)}$ "must" be useful!

Note conversely, that if Y has standard asymptotics in a sector which does not contain a separation ray, then Y is not uniquely determined eg $Y^{(1)}$ and $Y^{(1)}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$, $c \neq 0$,

both have standard asymptotics in the sector $0 < \arg z < \pi/3$.

~~and~~ $\cancel{Y^{(1)} = Y^{(2)} \begin{pmatrix} 1 & 0 \\ c_{21} & 1 \end{pmatrix} \neq Y^{(2)}}$

~~as~~ $\cancel{c_{21} \neq 0}$ ~~(why?)~~

<u>Exercise</u> Show that each $Y^{(i)}(z)$ has the same asymptotic expansion in its sector, i.e. for

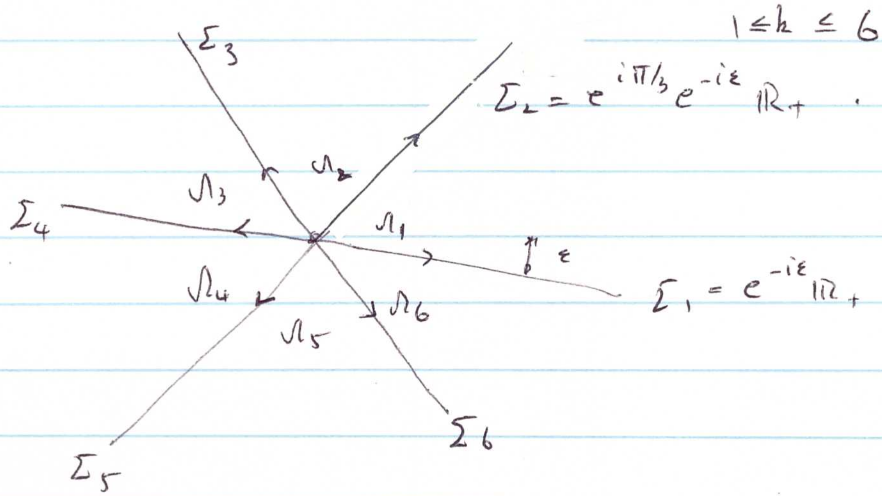
(134.0) $$Y^{(i)}(z) = \left(I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \cdots \right) e^{\theta \sigma_3}$$

Y_k, $k = 0, 1, 2, \ldots$, are are indep. of $Y^{(i)}$.

Now for some fixed small $\varepsilon > 0$, introduce the 6 regions $\Omega_1, \ldots, \Omega_6$ which are the components of $\mathbb{C} \setminus \Sigma$ when

(134.1) $$\Sigma = \bigcup_{k=1}^{6} \Sigma_k, \quad \Sigma_k = e^{i(k-1)\pi/3} e^{-i\varepsilon} \mathbb{R}_+,$$

$1 \leq k \leq 6$.



Σ is oriented outward as indicated.

Let

(135.1) $$Y(z) = Y^{(k)}(z), \quad z \in \Omega_k, \quad 1 \le k \le 6.$$

Note that in each of the <u>closed</u> sectors $\bar{\Omega}_k$, $Y(z)$ has standard asymptotics as in (134.0).

Now $\exists$ constant matrices $v_1, \dots, v_k$, indep. of z, st

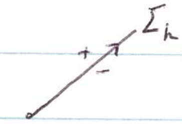
(135.2) $$Y^{(k)}(z) = Y^{(k-1)}(z)\, v_k, \quad k = 1, \dots, 6$$

where $Y^{(0)} \equiv Y^{(6)}$.

With the above notation, it is clear that $Y(z)$ solves the following RHP in the classical sense:

- $Y(z)$ is anal. in $\mathbb{C} \setminus \Sigma$, and cont. up to the bdry in each sector

(135.3)
- $Y_+(z) = Y_-(z)\, v(z), \quad z \in \Sigma / 0$



- $Y(z) e^{-\theta \sigma_3} \to I$ as $z \to \infty$ in each (closed) sector

where

(135.4) $$v(z) = v_k \quad \text{for } z \in \Sigma_k \setminus 0.$$

Now as $\operatorname{tr} L(z) = 0$, $\frac{\partial Y}{\partial z} = LY$, we have

$\det Y^{(k)}(z) = \text{constant}$ for each k. But $\det Y^{(k)}(z)$

$= \det \hat{Y}(z) \times \det e^{\theta\sigma_3} = \det \hat{Y}(z) \to 1$ as $z \to \infty$. Hence

$\det Y^{(k)}(z) \equiv 1 \quad \forall k, z \Rightarrow$

(136.1) $$\det v_k = 1$$

Standard arguments (<u>exercise</u>) $\Rightarrow$ $Y(z)$ is the unique solution of the RHP (135.3).

Consider $Y^{(1)} = Y^{(6)} v_1$. On the line

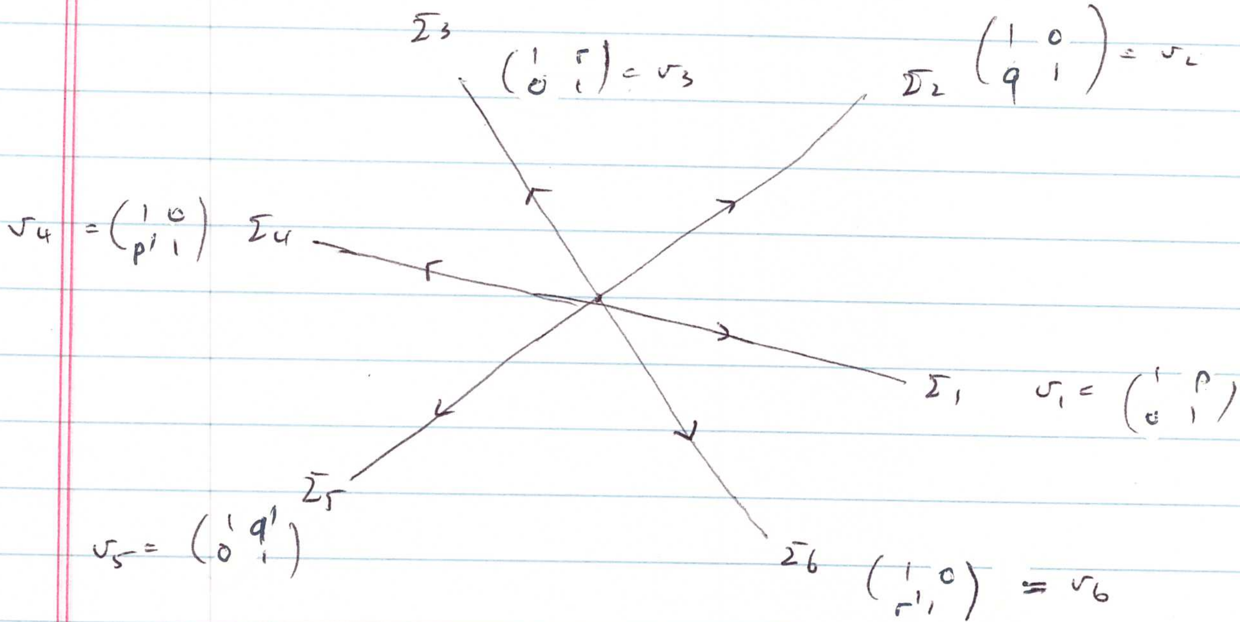
$\Sigma_1 = e^{-i\varepsilon}\mathbb{R}_+$, $\operatorname{Re}\theta \to -\infty$ as $z \to \infty$. Hence as

$$\hat{Y}^{(1)} = \hat{Y}^{(6)} e^{\theta\sigma_3} v_1 e^{-\theta\sigma_3}$$

$$= \hat{Y}^{(6)} \begin{pmatrix} v_{11} & v_{12} e^{2\theta} \\ v_{21} e^{-2\theta} & v_{22} \end{pmatrix}$$

and as $\hat{Y}^{(1)}(z)$, $\hat{Y}^{(2)}(z) \to I$ as $z \to \infty$, we must have $v_{11} = v_{22} = 1$, $v_{21} = 0$. Thus $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$ on Σ_1, for some constant p.

Similarly we find



Now observe from equation (117.2)

$$\frac{\partial Y}{\partial z} = LY = \begin{pmatrix} -4iz^2 - ix - 2iu^2 & 4uiz - 2w \\ -4uiz - 2w & 4iz^2 + ix + 2iu^2 \end{pmatrix} Y$$

that

(137.1) $$L^T(-z) = L(z)$$

Hence

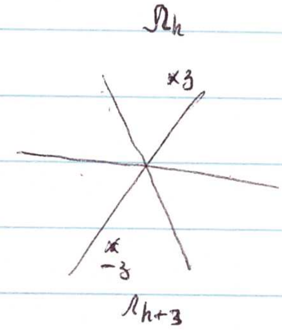
$$\frac{d}{dz}\left[Y(-z)^{-1}\right] = Y(-z)^{-1}\, Y'(-z)\, Y(-z)^{-1}$$
$$= Y(-z)^{-1}\, L(-z)\, Y(-z)\, Y(-z)^{-1}$$
$$= (Y(-z))^{-1}\, L(-z)$$

or

$$\frac{d}{dz}\left(Y(-z)^{-T}\right) = L(z)\left(Y(-z)\right)^{-T}$$

so that $Y(-z)^{-T}$ is a solution if $Y(z)$ is a solution.

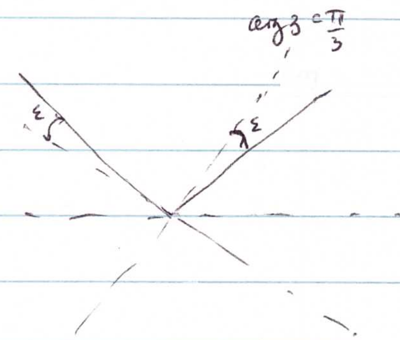
Thus for $z \in \Omega_k$, say,



$(Y^{(k+3)}(-z))^{-T}$ is a solution of $\frac{d\Psi}{dz} = L\Psi$

and as
$$(Y^{(k+3)}(-z))^{-T} = (\hat{Y}^{(k+3)}(-z))^{-T} \times e^{-\theta(-z)\sigma_3}$$
$$= (\hat{Y}^{(k+3)}(-z))^{-T} e^{\theta(z)\sigma_3}$$
$$= (I + O(\tfrac{1}{z}))\, e^{\theta(z)\sigma_3}, \quad z \to \infty$$

we see that $(Y^{(k+3)}(-z))^{-T}$ has standard asymptotics in Ω_k, and as Ω_k contains a separating ray



it follows by uniqueness that

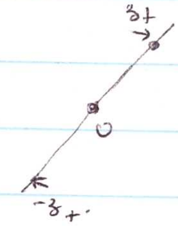
(138.1) $$(Y^{(k+3)}(-z))^{-T} = Y^{(k)}(z)$$

or

(138.2) $$Y(-z)^{-T} = Y(z), \quad z \in \mathbb{C} \setminus \Sigma.$$

Thus for $z \in \Sigma \setminus 0$, $Y_+(z) = Y_-(z)\, v(z) \Rightarrow$

$$Y_+(-z)^{-T} = Y_-(-z)^{-T} \, v(-z)^{-T}.$$



$$\Rightarrow \quad Y_+(z) = Y_-(z) \, v(-z)^{-T}$$

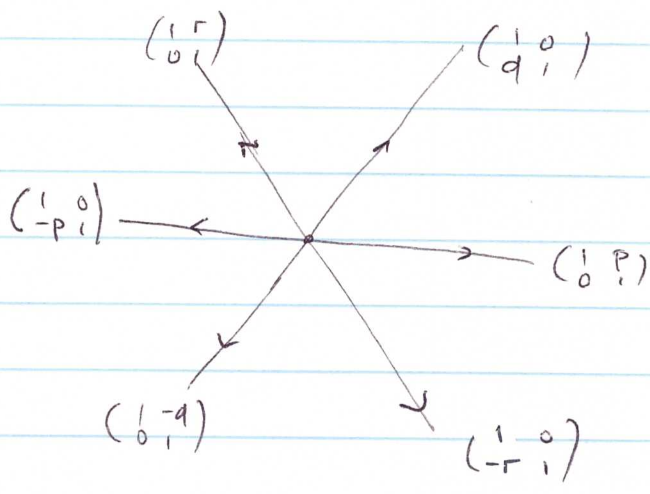
so that we conclude

(139.1) $$v(z) = v(-z)^{-T} \qquad , \quad z \in \Sigma$$

Hence

(139.2) $$p' = -p, \quad q' = -q, \quad r' = -r$$

Thus



Finally

$$Y^{(6)} = Y^{(5)} v_6 = Y^{(4)} v_5 v_6 \cdots = Y^{(1)} v_2 \cdots v_6$$
$$= Y^{(0)} v_1 v_2 \cdots v_6$$

$\Rightarrow$

(139.3) $$v_1 v_2 v_3 v_4 v_5 v_6 = I$$

Now

$$v_1 v_2 v_3 = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+pq & p \\ q & 1 \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+pq & p+r+pqr \\ q & 1+rq \end{pmatrix}$$

$$v_4 v_5 v_6 = \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -q \\ -p & 1+pq \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+rq & -q \\ -p-r-pqr & 1+pq \end{pmatrix}$$

Hence from (139.3) and $\det v_i = 1$, $i = 1, \dots, 6$,

$$\begin{pmatrix} 1+pq & p+r+pqr \\ q & 1+rq \end{pmatrix} = \begin{pmatrix} 1+pq & q \\ p+r+pqr & 1+rq \end{pmatrix}$$

ie

(140.1) $$q = p+r+pqr.$$

Thus the RHP is specified by points on the variety in $\mathbb{C}^3$

(140.2) $$V = \{(p,q,r) : q = p+r+pqr\}$$

The fact that V has "dimension" $= 2$, reflects the fact that Painlevé II is a 2nd order equation.
