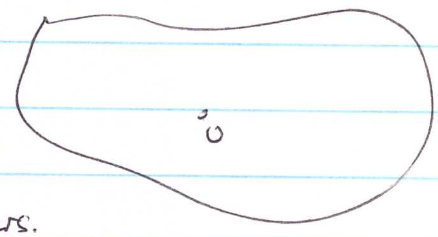
The following result establishes the general relationship between the operators $1 - C_w$ and matrix factorizations.

For definiteness, let Σ be a simple, bounded, rectifiable, A-regular closed contour in $\mathbb{C}$ with $z = 0$ in its interior.



Let $D(z)$ be the diagonal matrix

(173.0) $\begin{cases} D(z) = \operatorname{diag}(z^{k_1}, z^{k_2}, \dots, z^{k_n}) \\ \text{with } k_1 \geq k_2 \geq \dots \geq k_n, \text{ integers.} \end{cases}$

Let $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$. We say that $v = Gl(n, \mathbb{C})$, $v, v^{-1} \in L^\infty(\Sigma)$ has a <u>generalized (right-)standard factorization</u> relative to $L^p(\Sigma)$ in case

(173.1(i)) $\quad v = m_-^{-1} D(z) m_+ \quad$ on Σ

where

$$m_\pm \in A_\pm + \partial C(L_p)$$
$$m_\pm^{-1} \in B_\pm + \partial C(L_q)$$

and $\det A_\pm, \det B_\pm \neq 0$.

(173.1)(ii) The operator $X(\cdot) = (C^{+} \cdot m_+^{-1}) m_+ \in \mathcal{L}(L^p(\Sigma))$.

The integers $k_1 \geq \dots \geq k_n$ are called the (right-) partial indices of v.

Remark:

A similar defn. of generalized (left-)standard factorization, $v = m_+^{-1} D m_-$ etc., can of course be made, but by convention, by a gen. fact. we always mean a (right-)factorization.

The main theorem is the following.

Theorem 174.1

Let Σ, v be as above and $1 < p < \infty$. In order that the matrix function v admits a generalized factorization relative to $L^p(\Sigma)$ it is necessary and sufficient that the operator $1 - C_v = 1 - C^-(\cdot(v - I))$ is Fredholm in $L^p(\Sigma)$.

(175.1) Some properties of generalized factorizations: Let (m_+, m_-, D) be a gen. stand. factorization for v. Then

(i) the partial indices $k_1 \geq \dots \geq k_n$ are uniquely determined (the factors $m_\pm$ are not)

(ii) $$\text{index}\,(1 - C_v) = \sum_{i=1}^{n} k_i$$

(iii) $$\dim \ker (1 - C_v) = \sum_{k_i > 0} k_i$$

$$\dim \operatorname{coker} (I - C_v) = \sum_{k_i < 0} |k_i|$$

(iv) In general the k_i's are not stable under perturbation $v \to v + \delta v$. Only $\sum_{i}^{n} k_i = \text{index}\,(1 - C_v)$ is stable.

However, if $0 \leq k_i - k_{i+1} \leq 1$, $i = 1, \dots, n-1$, then the

k_i's are stable.

(v) The indices $\{k_i\}$ depend in general on p. Moreover $\text{ind}\,(1 - C_v)$ can also depend on p.

(vi) Th^m 157.1 is a special case of Theorem 174.1 with

$D(z) = I$.

(vii) If $v = v_-^{-1} v_+ = (I - w_-)^{-1}(I + w_+)$, $v_\pm, v_\pm^{-1} \in L^\infty(\Sigma)$ is a pt. wise factorization of v, then

(a) $1 - C_v$ is Fredholm $\Leftrightarrow$ $1 - C_w = 1 - (C^+ \cdot w_- + C^- \cdot w_+)$ is Fredholm

(b) $\dim \ker(1-C_v) = \dim \ker(1-C_w)$
$\dim \operatorname{coker}(1-C_v) = \dim \operatorname{coker}(1-C_w)$
so that if $(1-C_v)$, & hence $(1-C_w)$, is Fredholm, then

$$\operatorname{ind}(1-C_v) = \operatorname{ind}(1-C_w)$$

(viii) condition (173.1)(ii) cannot be omitted.

We will prove some of these facts: the remaining proofs can be found in Clancey-Gohberg or Spitkovsky-Litvinchuk.

(i) Suppose v has 2 gen. stan. factorizations

$$m_-^{-1} \Delta m_+ = \tilde{m}_-^{-1} \tilde{\Delta} \tilde{m}_+ .$$
$$\Delta = \operatorname{diag}(z^{k_1}, \dots, z^{k_n}), \quad \tilde{\Delta} = \operatorname{diag}(z^{\tilde{k}_1}, \dots, z^{\tilde{k}_n})$$
$$k_1 \geq k_2 \geq \dots \geq k_n \qquad \tilde{k}_1 \geq \tilde{k}_2 \geq \dots \geq \tilde{k}_n$$

(176.1) Then $$C_- \Delta = \tilde{\Delta} C_+$$

where $C_- = \tilde{m}_- m_-^{-1}$, $C_+ = \tilde{m}_+ m_+^{-1}$

By a familiar argument $C_\pm = E_\pm + C^\pm h$

where $\det E_\pm \neq 0$, and $h \in L^1 + L^p + L^q$.

From (176.1)

(177.1) $$(C_-)_{ij} = (C_+)_{ij}\, z^{\tilde{k}_i - k_j} \quad , \ 1 \le i, j \le n.$$

Now assume that for some ℓ, $1 \le \ell \le n$,

$\tilde{k}_\ell > k_\ell$ (the case $\tilde{k}_\ell < k_\ell$ is similar - exercise).

Then $\tilde{k}_i > k_j$ for $1 \le i \le \ell$, $\ell \le j \le n$.

It follows then from (177.1) that for
$1 \le i \le \ell$, $\ell \le j \le n$,

$$h_{ij}(z) \equiv \underbrace{(C_-)_{ij}(z)}_{\text{continuation of } C_- \text{ outside } \Sigma} \quad , \ z \text{ outside } \Sigma$$

$$\equiv \underbrace{(C_+)_{ij}}_{\text{contin. of } C_+ \text{ inside } \Sigma}(z)\, z^{\tilde{k}_i - k_j} \quad , \ z \text{ inside } \Sigma$$

is entire and bounded in $\mathbb{C}$. Hence $h_{ij}(z) = \text{const}_{ij}$

But as $\tilde{k}_i - k_j > 0$, const_{ij} must $= 0$.

It follows that $(C_-)_{ij} = 0$ and so C_- has the block form

$$C_- = \begin{array}{c c} & \begin{array}{cc} \ell-1 & n-\ell+1 \end{array} \\ \begin{array}{c} \ell \\ n-\ell \end{array} & \left(\begin{array}{c|c} \times & 0 \\ \hline \times & \times \end{array} \right) \end{array}$$

from which we see that $\quad \operatorname{rank} C_- \le l-1 + n-l \le n-1$.

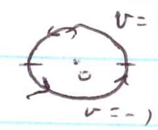
This is a contradiction as C_-^{-1} $\exists$ a.e. on Σ. Thus

$\theta(z) = \tilde{\theta}(z)$.

(ii) (iii) / (iv), see refs.

(v) Consider the following exple on $\Sigma = \{|z|=1\}$

$$v(z) = 1 \quad , z \in \Sigma \quad , z \in \mathbb{C}^+$$
$$\quad\;\; = -1 \quad , z \in \Sigma \quad , z \in \mathbb{C}^-$$



Seek

$$z^n m_+ = m_- v . \qquad m_\pm \in 1 + \partial C(L^p)$$
$$\text{and} \quad m_\pm^{-1} \in 1 + \partial C(L^q) \quad , \frac{1}{p} + \frac{1}{q} = 1.$$

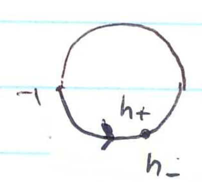
Let $\quad h(z) = \left(\dfrac{z-1}{z+1}\right)^{\frac{1}{2}}$ which is analytic in

$\mathbb{C} \setminus$ 

and $\quad h(z) \to 1$ as $z \to \infty$.

Then clearly $\quad h_+(z) = h_-(z)$ on Σ, $z \in \mathbb{C}^+$.

and $\quad h_+ = -h_-$ on Σ, $z \in \mathbb{C}^-$



Thus $\quad v = h_-^{-1} h_+$ on Σ.

Hence $\; l(z) \equiv z^n m_+ h_+^{-1} = m_- h_-^{-1}$

Assume first that $\boxed{p>2}$.

By familiar arguments $l(z)$ is analytic in $\mathbb{C} \setminus (\{1\} \cup \{-1\})$

But $h_\pm^{-1} = 1 + \partial C(L^s)$ for any $1<s<2$ and

so $m_\pm h_\pm^{-1} \in 1 + C^\pm(h)$ where $h \in L^p + L^s + L^r$

and $\frac{1}{r} = \frac{1}{p} + \frac{1}{s} < 1$, provided we choose, as we can,

s suff. close to 2. Thus for $p>2$, $l(z)$ is

analytic across Σ and bounded as $z \to \infty$.

Now if $n>0$. We see that $l(z)$ is anal. in $\mathbb{C}$,

and as it is bdd at ∞, it must be constant: but then

$n>0 \Rightarrow$ const $= 0 \Rightarrow m_\pm = 0$. Contradiction.

On the other hand if $n=0$, then $l(z) = \text{const} = 1$,

and so $m_\pm = h_\pm$ But $h_\pm \notin 1 + \partial C(L^p)$

for $p>2$. Again a contradiction.

So suppose $n = -\hat{n} < 0$.

Hence

$$\ell(z)\, z^{\hat n} = m_+ h_+^{-1} = z^{\hat n} m_- h_-^{-1} \qquad \text{and so}$$

$$\ell(z)\, z^{\hat n} = p(z) = z^{\hat n} + \cdots \qquad \text{is a monic polynomial.}$$

Thus
$$m_+ = p(z)\, h_+ = p(z) \left(\frac{z-1}{z+1}\right)_+^{-\frac{1}{2}}$$
$$m_- = p(z)\, h_-\, z^{-\hat n} = p(z) \left(\frac{z-1}{z+1}\right)_-^{\frac{1}{2}} z^{-\hat n}$$

Now as $m_\pm$ are invertible, as their extensions to $\{|z|<1\}$, $\{|z|>1\}$ resp. are invertible, it follows that $p(z)$ can only have zeros at $z = \pm 1$. If p had a zero at $z=1$, then as $z \to 1$ $m_+(z) \sim (z-1)^{\frac{1}{2}+m^\#}$ where $m^\# \geq 1$. and so $m_+^{-1} \sim (z-1)^{-(\frac{1}{2}+m^\#)} \notin L^q_{loc}$ for any $1<q<2$, $\frac{1}{q}+\frac{1}{p}=1$. Hence $p(z)$ cannot have a zero at $z=1$. and so $p(z) = (z+1)^{\hat n}$. Thus as $z \to -1$, $m_+^{-1} \sim (z+1)^{\frac{1}{2}-\hat n}$ which lies in L^q, $1<q<2$ if and only if $\hat n = 1$.

Thus $m_-^{-1} z^{-1} m_+ = v$ is a factorization for v ,

$$m_+ = (z+1)\left(\frac{z-1}{z+1}\right)_+^{\frac{1}{2}}$$

$$m_- = \frac{(z+1)}{z}\cdot\left(\frac{z-1}{z+1}\right)_-^{\frac{1}{2}}$$

$$m_\pm \in 1 + \partial C(L^p) \quad , (m_\pm)^{-1} \in 1 + \partial C(L^q)$$

Furthermore, it is an (important) exercise to show that indeed $X = (C^+ \cdot m_+^{-1})m_+ \in \mathcal{L}(L^p)$. Thus $(m_\pm, D = z^{-1})$ is a gen.-stand. fact. for v with index $= -1$.

(181.1) Exercise (a) Use the above methods to show that index $\neq -1$ for $1<p<2$

(b) Show that $1 - C_v$ is not Fredholm for $p=2$. Prove this by showing no fact. $z^n m_+ = m_- v$ $\exists$ with $m_\pm, m_\pm^{-1} \in 1 + \partial C(L^2)$. Alternatively show that $\dim \operatorname{coker}(1 - C_v) = \infty$ for $p = 2$.

(Hint: consider IRHP2$_{L^2}$ $M_+ = M_- v + F$ with rational F).

(vi) ✓

(vii) For $v = v_-^{-1} v_+ = (I - w_-)^{-1}(I + w_+)$

$$\begin{aligned}
(1 - C_w)h &= h - C^- h w_+ - C^+ h w_- \\
&= h - C^- h w_+ - C^- h w_- - h w_- \\
&= h v_- - C^- h(w_+ + w_-) = h v_- - C^- h(v_+ - v_-) \\
&= h v_- - C^-(h v_-)(v_-^{-1} v_+ - 1) = (1 - C^- \cdot (v - I))h v_-
\end{aligned}$$

$\Rightarrow$)

(182.1) $$(\mathbb{1} - C_\omega) = (\mathbb{1} - C_v) R_{v_-}$$

where $R_{v_-} h = h v_-$ = right mult. by v_-

(vii)(a)(b) now follow immediately from (182.1)

(viii) An example of a jump matrix v on a contour Σ with a factorization

$v = m_-^{-1} m_+$, where $m_\pm \in I + \partial(L^p)$, $m_\pm^{-1} \in I + \partial C(L^q)$

but $X = (C^+ \cdot m_+^{-1}) m_+ \notin \mathcal{L}(L^p(\Sigma))$ can be constructed

using some calculations in Spitkovsky – Litvinchuk pp 114-116.

In these pages the authors construct a function $k(\theta) \geq 0$

on the unit circle Σ with the following properties:

- $k \in L^p(\Sigma)$ $\forall\ 1 < p < \infty$ and $k^{-1} \in L^\infty(\Sigma)$.

- For the intervals $\gamma_s = \{e^{i\theta} : 2^{1-2s} - 1 \leq \frac{\theta}{\pi} < 2^{3-2s} - 1\}$ in Σ, $s = 1, 2, 3, \ldots$

(182.2) $$\frac{1}{|\gamma_s|} \left(\int_{\gamma_s} k(\theta)^p \, d\theta \right)^{\frac{1}{p}} \left(\int_{\gamma_s} k(\theta)^{-q} \, d\theta \right)^{\frac{1}{q}} = \frac{1}{3}(2+s^p)^{\frac{1}{p}} (2+s^{-q})^{\frac{1}{q}}$$

which goes to ∞ as $s \to \infty$.

Now a necessary and sufficient condition for $T = (C^+ \circ h^{-1}) h$ to be bded in $L^p(\Sigma)$ is that

$$\sup_\gamma \frac{1}{|\gamma|} \left(\int_\gamma h^p d\theta \right)^{\frac{1}{p}} \left(\int_\gamma h^{-q} d\theta \right)^{\frac{1}{q}} < \infty$$

where the sup is taken over all intervals in Σ.

It follows from (182.2) that T is not bded in any $L^p(\Sigma)$.

Now set

$$f(z) = e^{\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{e^{i\theta} + z}{e^{i\theta} - z} \right) \log h(\theta)\, d\theta}, \quad |z| < 1$$

Then a direct calculation shows that

$$|f_+(z)|^2 = h^2 \quad \text{ie} \quad |f_+(e^{i\theta})| = h(\theta).$$

Set

(183.1) $$v(z) = f_+(z) \Big/ \overline{f_+(z)}, \quad z \in \Sigma$$

Hence $v \in L^\infty$, $v^{-1} \in L^\infty$

Let
$$m_+(z) = f_+(z)$$
$$m_-(z) = \overline{f_+(z)} = \overline{f_+(\bar{z}^{-1})}$$

Then $v = m_-^{-1} m_+$ is a factorization of v

with $m_\pm \in c + \partial C(L^p)$, $m_\pm^{-1} \in c^{-1} + \partial C(L^q)$

for some $c>0$, by the above properties of k.

However, writing $m_+ = k e^{i\phi}$, we have for $h \in L^p(\Sigma)$,

$$Xh = (C^+ h m_+^{-1}) m_+ = (C^+ h e^{-i\phi} k^{-1}) k e^{i\phi}$$

$$= R_{e^{i\phi}}(T R_{e^{-i\phi}} h), \quad R_{e^{\pm i\phi}} h = h e^{\pm i\phi}$$

and so the unboundedness of $T \Rightarrow$ unbddedness of X.

Thus $v = f_+/\bar{f}_+$ has an "L^p, L^q" factorization

but not a gen. standard factorization.

Finally we illustrate some of the properties (i) ... (viii) with some 2×2 examples.

Let $\Sigma = \{|z| = 1\}$

For $a \neq 0$, set $v_a^{\uparrow}(z) = \begin{pmatrix} z & a \\ 0 & z^{-1} \end{pmatrix}$

Direct computation shows that

(185.1) $$v_a^{\uparrow} = \begin{pmatrix} z & a \\ 0 & z^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{az} & -\frac{1}{a} \end{pmatrix} \begin{pmatrix} z & a \\ 1 & 0 \end{pmatrix}$$

$$= \underbrace{-a^{-1}\begin{pmatrix} -\frac{1}{a} & 0 \\ -\frac{1}{az} & 1 \end{pmatrix}^{-1}}_{m_-^{-1}} \underbrace{\begin{pmatrix} z & a \\ 1 & 0 \end{pmatrix}}_{m_+}$$

$\delta_a^{\uparrow}(z) = I$ $\qquad \therefore k_1 = k_2 = 0$

(185.2) Again for $a \neq 0$, now set $v_a^{\downarrow}(z) = \begin{pmatrix} z & 0 \\ a & z^{-1} \end{pmatrix}$

Direct computation shows that

$$v_a^{\downarrow} = \begin{pmatrix} z & 0 \\ a & z^{-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ -\frac{a}{z} & 1 \end{pmatrix}^{-1}}_{m_-^{-1}} \underbrace{\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}}_{\delta_a^{\downarrow}(z)} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{m_+}$$

$\therefore k_1 = 1, k_2 = -1$

For $a = 0$

$$v_0 = v_0^{\uparrow} = v_0^{\downarrow} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = \underbrace{I^{-1}}_{m_-^{-1}} \underbrace{\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}}_{D_0(z)} \underbrace{I}_{m_+}$$

Thus we see that the k_i's are <u>not stable</u>: under perturbation

$$v_0 \to v_a^{\uparrow}, \quad D_0(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \to D_a^{\uparrow}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Of course, other perturbations preserve the k_i's:

$$D_a^{\downarrow} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = D_0$$

In both cases, however, $k_1 + k_2$ is preserved

i.e. the index is stable.

Now consider for $a \neq 0$

$$V_a^{\uparrow}(z) = \begin{pmatrix} z & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} I.$$

$$= m_-^{-1} \, D_a^{\uparrow}(z) \, m_+$$

$$V_a^{\downarrow}(z) = \begin{pmatrix} z & 0 \\ a & 1 \end{pmatrix} = \underbrace{I^{-1}}_{m_-^{-1}} \underbrace{\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}}_{D_a^{\downarrow}(z)} \underbrace{\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}}_{m_+}$$

And now in both cases

$$ {}_{v}\mathcal{D}_a^{\uparrow} = {}_{v}\mathcal{D}_a^{\downarrow} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} = {}_{v}\mathcal{D}_0 $$

when $V_0 = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} = I^{-1}\, {}_{v}\mathcal{D}_0\, I$, so the partial indices are stable under these perturbations.

The difference between the cases $v_a^{\uparrow}, v_a^{\downarrow}$ and $V_a^{\uparrow}, V_a^{\downarrow}$ is that $|k_1 - k_2| > 1$ for v_0 whereas $|k_1 - k_2| = 1$ for V_0 (cf (iv) p175).

<u>Exercise</u> Show that the k_i's are indeed stable for <u>all</u> perturbations $V_0 \to U$ of V_0.