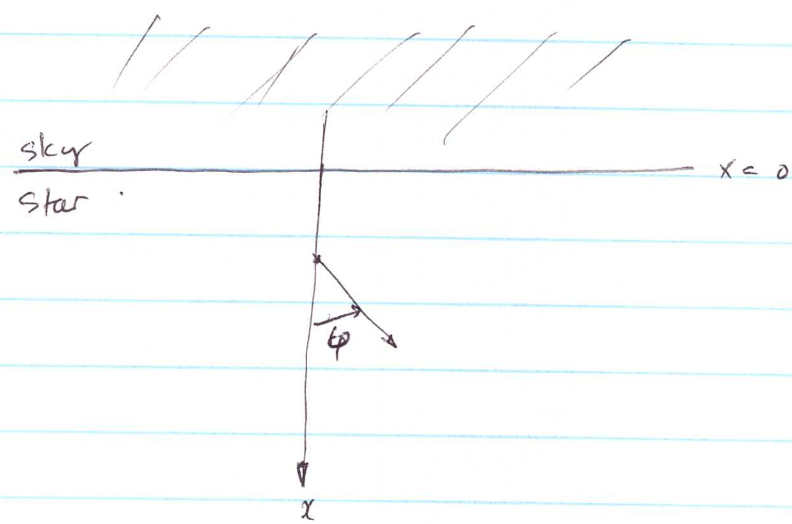
Lecture 13

We now show how the general Theorem 174.1 can be used to solve a problem in radiative equilibrium. (cf. H. Dym & H. McKean, Fourier Series and Integrals: Milne's equation §3.6, pp 176–184):

"Think of the star as so big that its curvature can be neglected and introduce as coordinates, the depth $0 \le x < \infty$ into the stellar interior and the inclination $0 \le \varphi \le \pi$ to the downward direction, as follows:



The distribution of radiation as regards depth and inclination is described by a "radiation density" $e = e(x, \varphi)$

so that the amount of radiation in a slab $a \le x \le b$ traveling at inclination $\alpha \le \varphi \le \beta$ is

$$\int_a^b dx \int_\alpha^\beta e(x,\varphi)\sin\varphi\, d\varphi$$

The equilibrium is produced and maintained by streaming (at speed 1, say) and by scattering, and a detailed balancing of these 2 mechanisms leads to (see ref to Hopf in D+McK) the following law:

$$\cos\varphi \frac{\partial e}{\partial x}(x,\varphi) + e(x,\varphi) = \frac{1}{2}\int_0^\pi e(x,\psi)\sin\psi\, d\psi$$

Milne's problem is to compute the angular distribution of radiation at the stellar surface

(189.1) $$I(\varphi) = \frac{e(0,\varphi)}{\frac{1}{2}\int_0^\pi e(0,\psi)\sin\psi\, d\psi}$$

under the condition that no radiation is coming in from the sky $e(0,\varphi) = 0$ for $0 \le \varphi \le \frac{\pi}{2}$, the so-called "law of darkening". To do this, one introduces the radiation intensity

$$f(x) = \frac{1}{2}\int_0^{\pi} e(x,\varphi)\sin\varphi\, d\varphi.$$

and solves

$$\frac{\partial e}{\partial x} + \sec\varphi\, e = e^{-x\sec\varphi}\frac{\partial}{\partial x}\left(e^{x\sec\varphi} e\right) = \sec\varphi\, f$$

for $\varphi < \pi/2$ and $\varphi > \frac{\pi}{2}$ separately, which leads to (the term at $x=\infty$ is absent)

$$(190.1)\quad \begin{cases} e(x,\varphi) = \sec\varphi \int_0^x e^{(y-x)\sec\varphi} f(y)\,dy, & 0<\varphi<\frac{\pi}{2} \\ = -\sec\varphi \int_x^{\infty} e^{(y-x)\sec\varphi} f(y)\,dy, & \frac{\pi}{2}<\varphi<\pi \end{cases}$$

We now obtain

$$f(x) = \frac{1}{2}\int_0^{\pi/2} e(x,\varphi)\sin\varphi\, d\varphi + \frac{1}{2}\int_{\pi/2}^{\pi} e(x,\varphi)\sin\varphi\, d\varphi.$$

$$= \frac{1}{2}\int_0^{\pi/2}\sec\varphi\left(\int_0^x e^{(y-x)\sec\varphi} f(y)\,dy\right)\sin\varphi\, d\varphi$$

$$+ \frac{1}{2}\int_{\pi/2}^{\pi}\left(-\sec\varphi\left(\int_x^{\infty} e^{(y-x)\sec\varphi} f(y)\,dy\right)\sin\varphi\, d\varphi\right.$$

$$= \frac{1}{2} \int_0^{\pi/2} \tan \psi \left(\int_0^x e^{(y-x)\sec\psi} f(y)\, dy \right) d\psi$$

$$+ \frac{1}{2} \int_{\frac{\pi}{2}}^{0} - \tan(\pi - \theta) \left(\int_x^\infty e^{(y-x)\sec(\pi-\theta)} f(y)\, dy \right) - d\theta .$$

$$= \frac{1}{2} \int_0^{\pi/2} \tan\psi \left(\int_0^\infty e^{-|y-x|\sec\psi} f(x)\, dx \right) d\psi .$$

$$= \int_0^\infty k(x-y)\, f(y) \qquad , x \geq 0$$

when

(191.1) $$k(x) = \frac{1}{2} \int_0^{\pi/2} e^{-|x|\sec\psi} \tan\psi \, d\psi .$$

$$= \frac{1}{2} \int_1^\infty e^{-|x| y} \frac{dy}{y}$$

Thus we are led to solving the following Wiener-Hopf problem

(191.2) $$f(x) = \int_0^\infty k(x-y)\, f(y)\, dy \qquad , x > 0$$

with k as above. Equation (191.2) is known as <u>Milne's equation</u>:

We want to show that (191.2) has non-trivial solutions, $f(x) \not\equiv 0$ ie we want to show that the homogeneous Wiener-Hopf eqtn has a solution.

Now clearly

(192.1) $$|k(x)| \le c\,\frac{e^{-|x|}}{|x|} \quad \text{as } |x| \to \infty$$

and $k(x)$ has a logarithmic singularity at $x = 0$. So in particular $k(x) \in L^1(\mathbb{R})$ and we can think of solving (191.2) in $L^2(\mathbb{R}_+)$.

By the calculations in Lectures 6 & 7, (191.2) is equivalent to a IRHP2$_{L^2}$ of the form (see 97.5)

$$m_+ = m_- v + H(z)$$

Here

$$m_+ = F(z) = \int_0^\infty e^{izx} f(x)\,\frac{dx}{\sqrt{2\pi}}.$$

$$H(z) = v^{-1}(z) \int_0^\infty e^{izx} g(x)\,\frac{dx}{\sqrt{2\pi}} = 0 \quad \text{as } g(x) = 0$$

(191.2 is homogen's)

$$v(z) = (1 - K(z))^{-1}, \quad \text{provided } K(z) \neq 1,$$

where

$$\mathbb{K}(z) = \int e^{izx} k(x)\,dx.$$

Thus we seek a solution $m_\pm \not\equiv 0$ to

(192.3) $$m_+ = m_- v, \quad m_\pm \in \partial C(L^2)$$

Now

$$K(z) = \frac{1}{2} \int_{\mathbb{R}} e^{izx} \int_1^\infty e^{-|x|y} \frac{dy}{y}.$$

$$= \frac{1}{2} \int_1^\infty \frac{dy}{y} \int_{\mathbb{R}} e^{izx - |x|y} \, dx.$$

$$= \frac{1}{2} \int_1^\infty \frac{dy}{y} \left(\int_0^\infty e^{(iz-y)x} \, dx + \int_{-\infty}^0 e^{(iz+y)x} \, dx \right)$$

$$= \frac{1}{2} \int_1^\infty \frac{dy}{y} \left(-\frac{1}{iz-y} + \frac{1}{iz+y} \right)$$

$$= \frac{1}{2} \int_1^\infty \frac{dy}{y} \frac{-iz-y+iz-y}{-(z^2+y^2)} = \int_1^\infty \frac{dy}{y^2+z^2}$$

$$= \int_0^1 \frac{du}{1+u^2z^2}.$$

Now observe that $K(1) = 1$ and so

$v(z) = 1 - K(z) = 0$ as $z = 1$. This means that

IR1+R2 above is singular. To remedy the problem we must introduce more smoothness on $m_\pm$, or equivalently we must $f(x)$ decay faster ie $L^2(\mathbb{R}_+)$ is not big enough to contain our solution.

To see how to do this, let $0<\rho<1$

and set

(194.1) $$\tilde{f}(x) = f(x)e^{-\rho x}$$

Then (191.2) takes the form

(194.2) $$\tilde{f}(x) = \int_0^\infty \tilde{k}(x-y)\,\tilde{f}(y)\,dy$$

where

$$\tilde{k}(x) = e^{-\rho x}\,k(x)$$

and we have preserved the Wiener-Hopf form of the equation. and also, from (192.1) we see that $\tilde{k}$ is still in $L^1(\mathbb{R})$. We must now show that the IRHP2$_{L^2}$

(194.3) $$\tilde{m}_+ = \tilde{m}_-\,\tilde{v}(z)$$

has a non-trivial solution $\tilde{m}_\pm \in \partial C(L^2)$

where $\tilde{v}(z) = (1-\hat{\tilde{k}}(z))^{-1}$, provided $\hat{\tilde{k}}(z) \neq 1$.

Now

$$\tilde{K}(z) = \frac{1}{2}\int_{\mathbb{R}} e^{izx} e^{-px} \int_1^\infty e^{-|x|y} \frac{dy}{y}$$

$$= \int_0^1 \frac{du}{1+u^2(z+ip)^2}$$

and so for $z \in \mathbb{R}$

$$1-\tilde{K}(z) = \int_0^1 \frac{u^2(z+ip)^2}{1+u^2(z+ip)^2}\, du$$

$$= \int_0^1 \frac{u^2(z+ip)^2(1+u^2(z-ip)^2)\, du}{|1+u^2(z+ip)^2|^2}$$

$$= \int_0^1 \frac{u^2\left[(z+ip)^2 + u^2(z^2+p^2)^2\right] du}{|1+u^2(z+ip)^2|^2}.$$

$$= \int_0^1 \frac{u^2\left[(z^2-p^2 + u^2(z^2+p^2)^2) + 2ipz\right]}{|1+u^2(z+ip)^2|^2}$$

(195.1) ie $$1-\tilde{K}(z) = \int_0^1 \frac{u^2(z^2-p^2+u^2(z^2+p^2)^2)}{|1+u^2(z+ip)^2|^2}$$

$$+ 2ipz \int_0^1 \frac{u^2\, du}{|1+u^2(z+ip)^2|^2}$$

[Note that

$$1+u^2(z+i\rho)^2=0 \iff z+i\rho = \pm\frac{i}{u}.$$

$$\iff z = i\left(\frac{\pm 1}{u} - \rho\right).$$

As $1/u \geq 1$ and $\rho < 1$, we see that

$$1+u^2(z+i\rho)^2 \neq 0 \quad \text{for } z\in\mathbb{R}, \quad u\in[0,1].$$

Thus $\tilde{k}(z)$ has no singularities on $\mathbb{R}$: this is of course obvious a priori, as $\tilde{k} \in L^1(\mathbb{R})$].

From (195.1) we see that $1-\tilde{k}(z)=0 \Rightarrow$ in particular that $\operatorname{Im}(1-\tilde{k}(z))=0$ and so $z=0$.

But if $z=0$.

$$\operatorname{Re}(1-\tilde{k}(0)) = \int_0^1 \frac{u^2(-\rho^2+u^2\rho^4)}{|1+u^2(i\rho)^2|^2}$$

$$= -\rho^2\int_0^1 \frac{u^2(1-\rho^2u^2)}{|1-u^2\rho^2|^2} < 0$$

Thus

$$1-\tilde{k}(z) \neq 0$$

and so $\tilde{v}(z)$ is non-singular.

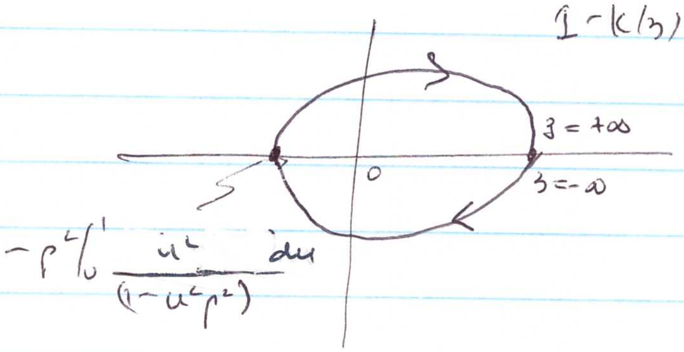
We now compute the winding # of $1 - k(z)$ as z goes from $-\infty$ to $+\infty$.

As $z \to \pm\infty$ $\quad 1 - \tilde{k}(z) \to 1$

But for $z < 0$, we see that $\operatorname{Im} 1 - \tilde{k}(z) < 0$ and as as z increases from $-\infty$, $1 - \tilde{k}(z)$ stays in $\mathbb{C}^-$ until $z = 0$. At $z = 0$, as shown above,

$$\operatorname{Re}\left(1 - k(0)\right) < 0$$

And then for $z > 0$, $\operatorname{Im} 1 - \tilde{k}(z) > 0$. Thus we have



Trajectory of $1 - \tilde{k}(z) \Big|_{-\infty}^{\infty}$.

Hence ~~wind~~

$$\text{winding \# of } \left(1 - \tilde{k}(z)\right) = -1$$

so

$$\text{winding \# of } \tilde{v}(z) = (1-\tilde{k}(z))^{-1} = +1.$$

By the general Theorem 174.1 this implies

(transformed from $\Sigma = \{|z|=1\}$ to $\Sigma = \mathbb{R}$)

that

$$\dim \ker (1 - C_{\tilde{v}}) = 1$$

and so there is a 1-dimensional family of

solutions $\tilde{m}_+ = \tilde{m}_- \tilde{v}$, $\tilde{m}_\pm \in \partial C(L^2)$.

(*) $\boxed{\text{Insert 198+, 198++}}$

This then shows that (for any $0<\rho<1$) the Milne equation

has as solution f st $f e^{-\rho x} \in L^2(\mathbb{R}_+)$.

A priori f depends on ρ : but if $0<\rho_1<\rho_2<1$

and $f_1 e^{-\rho_1 x}$ and $f_2 e^{-\rho_2 x}$ are the solutions

obtained by the above procedure, then clearly

Insert (*) on p198 To compute $\tilde{m}_\pm$ we proceed as

follows: set $v^\# \equiv \frac{z+i\rho}{z-i\rho}\tilde{v}$. As winding # $\left(\frac{z+i\rho}{z-i\rho}\right)$ $= -1$, this implies winding # of $v^\# = 0$. Thus $\log v^\#(z) \to 0$ as $z \to \pm\infty$ and we can factorize $v^\#$ as $m_+^\# = m_-^\# v^\#$, $m^\# \to 1$ as $|z| \to \infty$ using the explicit formula

$$m^\# = e^{\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\log v^\#(s)}{s-z}\,\frac{ds}{2\pi i}}$$

Now

$$(198+.1) \qquad \tilde{m}_+ = \tilde{m}_- \frac{z-i\rho}{z+i\rho} v^\# = \frac{\tilde{m}_-}{m_-^\#}\cdot\frac{z-i\rho}{z+i\rho}\cdot m_+^\#$$

Set

$$h(z) \equiv \frac{\tilde{m}(z)}{m^\#(z)}(z+i\rho), \quad \operatorname{Im} z > 0$$

$$\equiv \frac{\tilde{m}(z)}{m^\#(z)}(z-i\rho), \quad \operatorname{Im} z < 0.$$

By (198+.1), $h(z)$ is continuous across $\mathbb{R}$ and hence $h(z)$ is entire. As $\tilde{m}(z) \to 0$ as $z \to \infty$, we see that $h(z) = o(z)$ as $z \to \infty$. Thus $\Rightarrow$ $h(z) \equiv c$

= constant . Thus

$$(198\text{++}.1) \qquad \tilde{m}_{\pm}(z) = c\,\frac{m^{\#}_{\pm}(z)}{z \pm i\rho}, \quad z \in \mathbb{R}.$$

which $\Rightarrow$

$$f(x) = e^{\rho x} \int_{-\infty}^{\infty} e^{-izx}\, \tilde{m}_{+}(z)\, \frac{dz}{2\pi i}, \quad x > 0$$

To compute $e(0,\psi)$, $\pi/2 < \psi < \pi$, we 190.1.

$$e(0,\psi) = -\sec\psi \int_0^\infty e^{y \sec\psi} f(y)\, dy$$

$$= -\sec\psi \int_0^\infty e^{y(\sec\psi + \rho)} \tilde{f}(y)\, dy.$$

$$= -\sec\psi \int_0^\infty e^{iy(-i(\sec\psi + \rho))} \tilde{f}(y)\, dy.$$

$$= -\sec\psi \sqrt{2\pi}\; \tilde{m}\left(-i(\sec\psi + \rho)\right) \qquad (\text{note: } -i(\sec\psi+\rho) \in i\mathbb{R}_+)$$

$$= \underset{\uparrow\ (198\text{++}.1)}{\frac{c'\sec\psi}{\left(-i(\sec\psi+\rho) + i\rho\right)}}\; e^{\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\log v^{\#}(s)}{s - (-i(\sec\psi+\rho))}\frac{ds}{2\pi i}}.$$

where
$$v^{\#}(s) = \frac{s+i\rho}{s-i\rho}\; \frac{1}{(s+i\rho)^2 \int_0^1 \frac{u^2\, du}{1+u^2(s+i\rho)^2}}$$

Thus

$$(198\text{++}.2) \quad e(0,\psi) = c''\, e^{\frac{-1}{2\pi i}\int_{\mathbb{R}} \frac{\log\left[(s^2+\rho^2)\left(\int_0^1 u^2/_{1+u^2(s+i\rho)^2}\, du\right)\right] ds}{s + i\rho + i\sec\psi}}, \quad \frac{\pi}{2} < \psi < \pi.$$

(Why is RHS indep of $0<\rho<1$?) . (cf 198++.2 with Sym-Mck, p183)

$f_1 e^{-\rho_2 x} = f_1 e^{-\rho_1 x} e^{-(\rho_2-\rho_1)x}$ solves the

same equation as $f_2 e^{-\rho_2 x}$ and $f_1 e^{-\rho_2 x} \in L^2(0,\infty)$.

As $\tilde{m}_+ = \tilde{m}_- \tilde{v}$, $\tilde{m}_\pm \in \partial C(L^2)$ has a 1-parameter

family of solutions for ρ_2, it follows that $f_1 e^{-\rho_2 x} = f_2 e^{-\rho_2 x}$

ie $f_1 = f_2$.

For more information on the problem, particularly the computation of $I(\varphi)$ in (189.1), see Dym + McKean

and the references therein

(+) Insert 199+

We now begin our analysis of the

asymptotics of some RHP's.

First we consider the Szegö Strong Limit Theorem

(see Lecture 6).

For $\varphi(z) > 0$, $\varphi \in L^1(\Sigma)$, $\Sigma = \{|z| = 1\}$, let

$$D_n = \det T_n = \det (\varphi_{j-k})_{0 \le j,k \le n}$$

Insert on p 199(+)

Exercise: (Dym + McK p178 et seq). A model problem for Milne's equation:

Consider the equation

$$(199+.1) \qquad f(x) = \frac{1}{2}\int_0^\infty e^{-|x-y|} f(y)\, dy \; , \qquad x>0$$

By differenting twice show that $f(x) = c(1+x)$ are the only solutions of (199+.1). Obtain these solutions by utilizing the Wiener-Hopf technique as in Milne's equation above.

be the associated Toeplitz determinant, where

$$\varphi_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}\varphi(e^{i\theta})\,d\theta \quad , \quad m \in \mathbb{Z}$$

Then if $\varphi = e^{L(z)} \in L^1(\Sigma)$ and $\sum_{k=1}^{\infty} k|L_k|^2 < \infty$,

then as $n \to \infty$

$$D_n = e^{(n+1)L_0 + \sum_1^{\infty} k|L_k|^2}(1+o(1))$$

where $L_m = \int_\Sigma e^{-im\theta} L(e^{i\theta})\frac{d\theta}{2\pi}$, $m \in \mathbb{Z}$

Szegö proved this result originally with much stronger assumptions on φ. We will prove the theorem under the condition that

- $\varphi(z) > 0$ on Σ
- $\varphi(z)$ is analytic in an annulus $\{\rho < |z| < \frac{1}{\rho}\} \supset \Sigma$ for some $0<\rho<1$.

Our purpose is to illustrate the steepest-descent method is one of the simplest RH situations.

Recall from (91.2) (see also P. Deift "Integrable Operators AMS Trans (2) 189, 1999 p69-84).

we have the formula,

(201.1)
$$\log D_n = - \int_0^1 \left[\int_\Sigma \left(\sum_{j=1}^{2} F_{t,j}'(z)\, G_{t,j}(z) \right) dz \right] \frac{dt}{t} .$$

Here

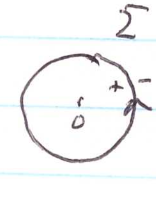
(201.2)
$$\begin{pmatrix} F_{t,1} \\ F_{t,2} \end{pmatrix} = m_{t+} \begin{pmatrix} z^{n+1} \\ 1 \end{pmatrix}$$

(201.3)
$$\begin{pmatrix} G_{t,1} \\ G_{t,2} \end{pmatrix} = (m_{t,+}^T)^{-1} \begin{pmatrix} z^{-(n+1)}(1-\varphi_t)/2\pi i \\ -(1-\varphi_t)/2\pi i \end{pmatrix}$$

$$\varphi_t = (1-t) + t\,\varphi(z), \quad z \in \Sigma, \quad 0 \le t \le 1.$$

and $m_{t,\pm}$ solves the normalized RHP (Σ, v_t)

where
$$v_t = \begin{pmatrix} \varphi_t & -(\varphi_t - 1) z^{n+1} \\ z^{-(n+1)}(\varphi_t - 1) & (2-\varphi_t) \end{pmatrix}$$



The idea of the proof is to move the z^{n+1} term in v_t into $|z|<1$ and the $z^{-(n+1)}$ term in v_t into $|z|>1$: Then as $n\to\infty$, these terms are exponentially small.

But first we must separate these terms algebraically. This is done using the lower-upper pt. wise factorization of v_t:

$$v_t = \begin{pmatrix} 1 & 0 \\ z^{-(n+1)}(1-\varphi_t^{-1}) & 1 \end{pmatrix} \begin{pmatrix} \varphi_t & 0 \\ 0 & \varphi_t^{-1} \end{pmatrix} \begin{pmatrix} 1 & -(1-\varphi_t^{-1}) z^{n+1} \\ 0 & 1 \end{pmatrix}$$

which is easily verified.

We now utilize the analyticity of $\varphi(z)$ in $\{\rho<|z|<\rho^{-1}\}$. As $\varphi(z)>0$ on Σ, we see that on Σ $\varphi_t = (1-t)+t\,\varphi(z) \geq c>0$ where $c = \min_z(1,\varphi(z))>0$

Hence $\exists$ $0<\rho<\rho^{(1)}<1$ such that

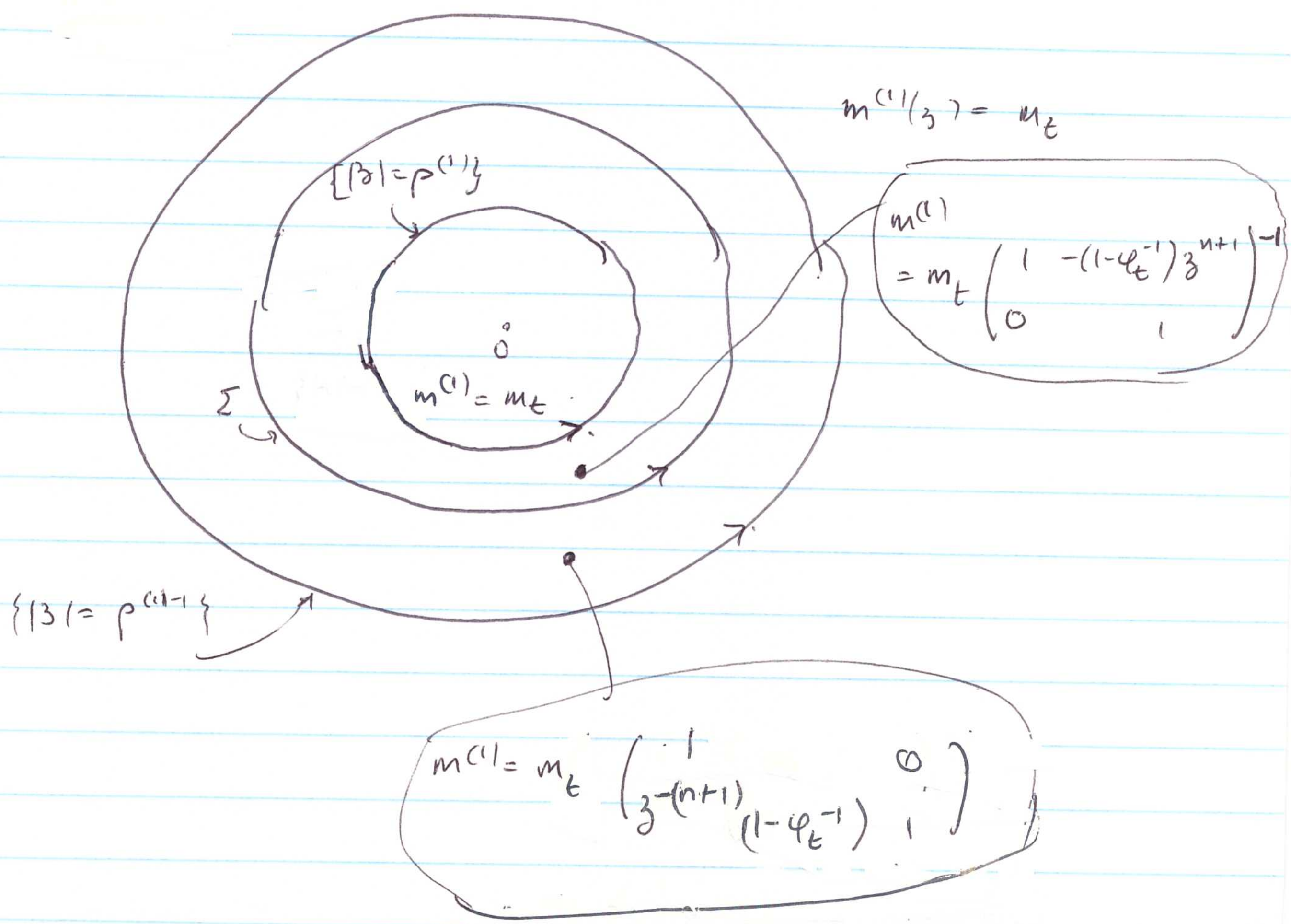
$\varphi_t(z)$ is invertible (and analytic) in a nbhood of

$$\rho < \rho^{(1)} < |z| < \rho^{(1)-1} < \rho^{-1}$$

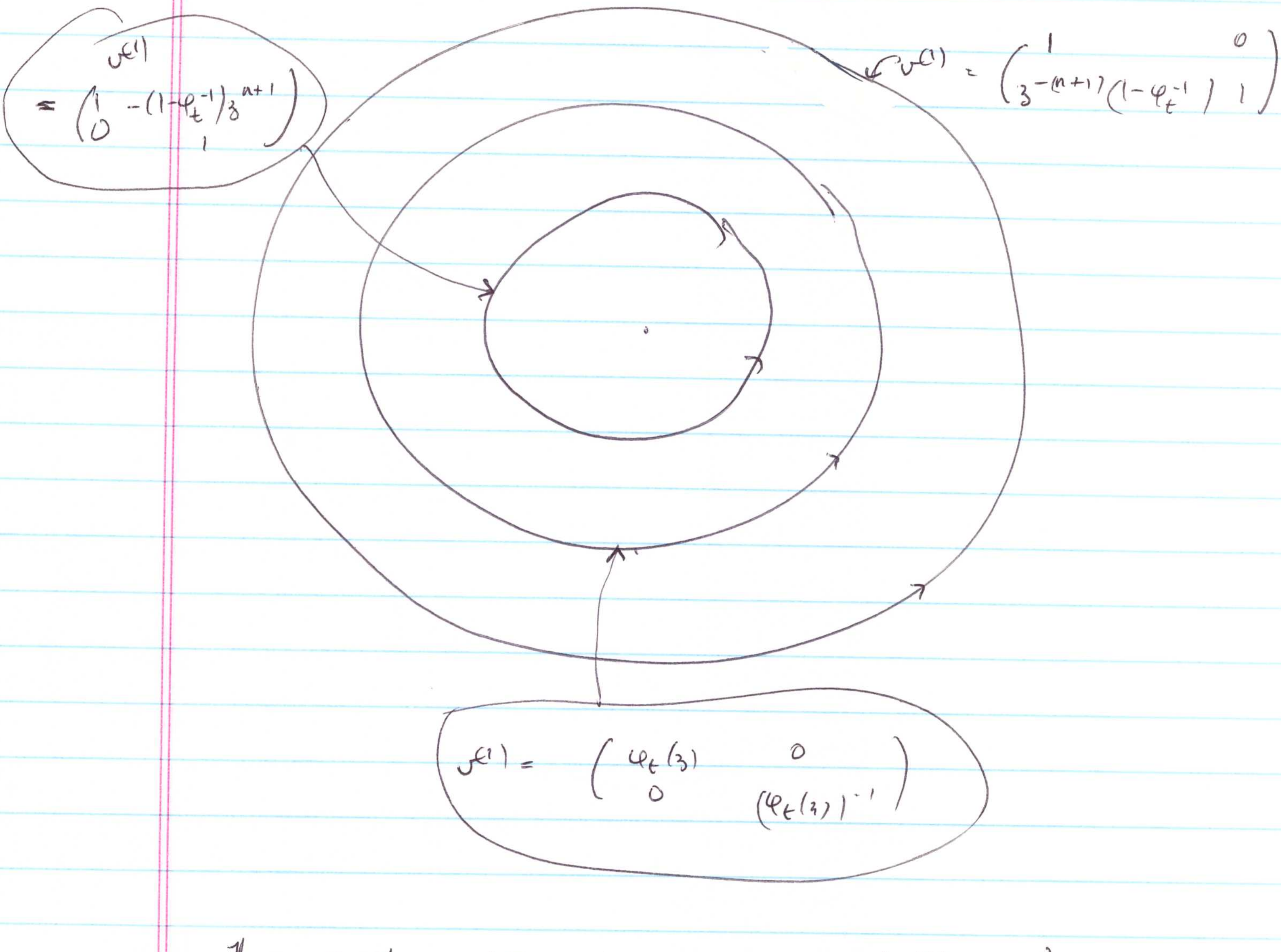
Extend Σ to a union of 3 circles.

$$\Sigma^{(1)} = \{|z| = \rho^{(1)}\} \cup \Sigma \cup \{|z| = (\rho^{(1)})^{-1}\}$$

all oriented counter-clockwise. Define $m^{(1)}$ as a piece-wise analytic function as follows:



A straightforward calculation shows that $m^{(1)}$ solves the normalized RHP $(\Sigma^{(1)}, v^{(1)})$ where



The RHP's (Σ, v_t) and $(\Sigma^{(1)}, v^{(1)})$ are clearly equivalent: the solution of one can be obtained from the other by simple algebraic operations, and vice versa.

Now observe that as $n \to \infty$, $v^{(1)}(z) - I$ goes to 0 uniformly (and exponentially) for $z \in \{|z| = \rho^{(1)}\} \cup \{|z| = \rho^{(1)-1}\}$ (and also uniformly for $t \in [0,1]$). In other words $v^{(1)}(z)$ converges uniformly on $\Sigma^{(1)}$ to the jump matrix

$$v^{\infty}(z) = I \qquad , |z| = \rho^{(1)} \text{ or } |z| = (\rho^{(1)})^{-1}$$

$$= \begin{pmatrix} \varphi_t & 0 \\ 0 & \varphi_t^{-1} \end{pmatrix} \quad \text{on } \Sigma$$

It then follows that if

$$(1 - C_{v^{(1)}})\,\mu^{(1)} = I$$

$$(1 - C_{v^{\infty}})\,\mu^{(\infty)} = I$$

are the respective associated integral operators

then $\mu^{(1)} = \dfrac{1}{1 - C_{v^{(1)}}} I = \dfrac{1}{1 - C_{v^{\infty}}} I + \dfrac{1}{I - C_{v^{(1)}}}\left(C_{v^{(1)}} - C_{v^{\infty}}\right) (1 - C_{v^{\infty}})^{-1} I$

$$= \mu^{(\infty)} + \underbrace{\frac{1}{1-C_{v^{(1)}}}\left(C^-\left(v^{(1)}-v^{\infty}\right)\right)\mu^{\infty}}_{O\left(e^{-\gamma n},\ \text{some } \gamma>0\right)}.$$

i.e.

(206.1) $$\|\mu^{(1)} - \mu^{\infty}\|_{L^2(\Sigma^{(1)})} = O(e^{-\gamma n}) \quad \text{as } n\to\infty.$$

It is an important exercise that in fact

(206.2) $$\|\mu^{(1)} - \mu^{\infty}\|_{L^\infty(\Sigma^{(1)})} = O(e^{-\gamma n})$$

The solution of the normalized RHP

$(\Sigma^{(1)}, v^{\infty}) \equiv (\Sigma, v^{\infty})$ is given by

$$m_\infty = \begin{pmatrix} e^{g_t} & 0 \\ 0 & e^{-g_t} \end{pmatrix}$$

where $g_t = \int_\Sigma \frac{\log \varphi_t(s)}{s-z}\frac{ds}{2\pi i}$.

It follows in particular from (206.2) that on Σ

$$m_\pm^{(1)}(z) \to m_\pm^{(\infty)}(z)$$

uniformly for $z \in \Sigma$. Moreover (exercise) the derivatives also converge

$$\frac{d}{dz} m_{\pm}^{(1)}(z) \to \frac{d}{dz} m^{(\infty)}(z), \qquad n \to \infty$$

uniformly for $z \in \Sigma$.

We see in particular from the relation between $m^{(1)}(z)$ and $m_t(z)$ in $\rho^{(1)} < |z| < 1$,

$$m_t(z) = m^{(1)} \begin{pmatrix} 1 & -(1-\varphi_t^{-1}) z^{n+1} \\ 0 & 1 \end{pmatrix}$$

that

$$m_{t+}(z) \sim m_+^{\infty} \begin{pmatrix} 1 & -(1-\varphi_t^{-1}) z^{n+1} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{(g_t)_+} & 0 \\ 0 & e^{-(g_t)_+} \end{pmatrix} \begin{pmatrix} 1 & -(1-\varphi_t^{-1}) z^{n+1} \\ 0 & 1 \end{pmatrix}$$

uniformly for $z \in \Sigma$, $0 \le t \le 1$.

Inserting this asymptotic formula into (201.1)(201.2)(201.3) we find (exercise).

$$\log D_n \sim -1 \int_0^1 dt \int_\Sigma (1-\varphi(z)) \Big(2\varphi_t^{-1} g'_{t,+} - \varphi_t' \varphi_t^{-2} + \frac{n+1}{z} \varphi_t^{-1} \Big) \frac{dz}{2\pi i}$$

Now from

$$g'_{t+} = g'_{t-} + (\log \varphi_t)'$$

and hence

$$2\varphi_t^{-1} g'_{t,+} - \varphi_t' \varphi_t^{-2} = \varphi_t^{-1} (g'_{t+} + g'_{t-}),$$

we have

$$\log D_n \sim - \int_0^1 dt \int_\Sigma (1-\varphi(z)) \Big(\varphi_t^{-1} (g'_{t+} + g'_{t-}) + \frac{n+1}{z} \varphi_t^{-1} \Big) \times \frac{dz}{2\pi i}$$

$$= (n+1) L(\varphi) + \int_0^1 dt \int_\Sigma (\varphi - 1) \varphi_t^{-1} (g'_{t+} + g'_{t-}) \frac{dz}{2\pi i}$$

as $\int_0^1 dt \int_\Sigma (1-\varphi) \frac{1}{1+t(\varphi-1)} \frac{d\theta}{2\pi}$

$$= - \int_\Sigma \int_0^1 \frac{d}{dt} \log (1+t(\varphi-1)) \frac{d\theta}{2\pi} = -L(\varphi)$$

But for $\varepsilon > 0$ small we have the elementary integration identity,

$$\int_0^1 dt \int_{\{|z| = 1-\varepsilon\}} dz \, \frac{\varphi(z) - 1}{\varphi_t(z)} \int_{\{|s|=1\}} \frac{\log \varphi_t(s)}{(s-z)^2}.$$

$$= \int_{\{|z| = 1-\varepsilon\}} dz \int_{\{|s|=1\}} ds \, \frac{\log \varphi(z) \log \varphi(s)}{(s-z)^2}.$$

$$- \int_0^1 dt \int_{|s|=1} ds \, \frac{\varphi(s) - 1}{\varphi_t(s)} \int_{(|z| = 1-\varepsilon)} dz \, \frac{\log \varphi_t(s)}{(s-z)^2}.$$

Letting $\varepsilon \downarrow 0$, we learn that

$$\int_0^1 dt \int_\Sigma (\varphi(z) - 1) \varphi_t^{-1}(z) \, (g_{t+}' + g_{t-}') \frac{dz}{2\pi i}$$

$$= \int_\Sigma \log \varphi(z) \left(\frac{d}{dz} \left(\int \frac{\log \varphi(s)}{s-z} \right) \right)_+ .$$

$$= \sum_{k=1}^{\infty} k |L_k|^2 .$$

This proves the strong Szegö limit theorem for analytic $\varphi(z)$. $\square$