Lecture 14

In this lecture we compute

(1) the long-time behavior of solutions of the defocusing NLS equation

(2) the asymptotics of orthogonal polynomials with exponential weights $e^{-V(x)}dx$, $V(x)>0$.

(1) (Ref: D-Zhou, Long-time asymptotics, ... CPAM (2003))

Let $u(x,t)$ solve NLS

$$iu_t + u_{xx} - 2|u|^2 u = 0 \quad , \quad -\infty < x < \infty ,$$

with $u(x,0) = u_0(x) \in H^{1,1} = \{f \in L^2 : f', xf \in L^2\}$.

Such (weak, global) solutions $\exists$ and are unique.

Let $r(z)$ be the reflection coeff (see pp 102 et seq. and above refer.) for the assoc. Lax linear operator with $u = u_0$.

We have $r \in H_1^{1,1} = H^{1,1} \cap \{\|r\|_\infty < 1\}$. Set

(210.1) $$\nu(z) = -\frac{1}{2\pi}\log(1-|r(z)|^2) \quad , \quad |\alpha(z)|^2 = \frac{\nu(z)}{2}$$

and

(210.2) $$\arg \alpha(z) = \frac{1}{\pi}\int_{-\infty}^{z} \log(z-s)\, d\left(\log(1-|r(s)|^2\right) + \frac{\pi}{4} + \arg\Gamma(i\nu(z)) + \arg r(z)$$

Let $0 < \kappa < \frac{1}{4}$ and $z_0 = x/2t$ = stat. phase pt.

We shall show how to prove that as $t \to \infty$

(211.1) $$u(x,t) = u_{as}(x,t) + O\left(\frac{1}{t^{\frac{1}{2}+\kappa}}\right)$$

where

(211.2) $$u_{as}(x,t) = \frac{1}{t^{1/2}} \alpha(z_0) e^{ix^2/4t - i\nu(z_0)\log 2t}$$

- Above asymp. form originally due to Zakharov + Manakov, but without error estimates
- if we allow more smoothness and decay on $u_0(x)$ we can improve the ~~error~~ error estimate to $O\left(\frac{\log t}{t}\right)$.

Let $\theta = xz - tz^2$. We prove (211.1) utilizing the RHP ~~for~~ NLS:

Let $$v_\theta(z) = \begin{pmatrix} 1-|r(z)|^2 & re^{i\theta} \\ -\bar{r}e^{-i\theta} & 1 \end{pmatrix}$$

and let $m_\pm = m_\pm(z; x, t) \in I + \partial C(L^2)$ be the
$\uparrow\uparrow$
x, t fixed

solution of the normalized RHP (Σ, v_θ) where

$\Sigma = \mathbb{R}$, $\xrightarrow[\mathbb{R}]{} $ $-\infty \to \infty$.

If

$$m(z) = m(z; x, t) = I + \frac{m_1(x,t)}{z} + o\left(\frac{1}{z}\right)$$

as $z \to \infty$, then

(212.1). $$u(x,t) = -i\,(m_1(x,t))_{12}$$

The idea of the proof is to examine

$\operatorname{Re} i\theta$: We would like to move $e^{i\theta}$ into

regions of the complex plane where $\operatorname{Re} i\theta$ is <u>decreasing</u>.

Note that

$$\theta = xz - tz^2 = -t(z-z_0)^2 + tz_0^2.$$

and so

$$\operatorname{Re} i\theta = -t \operatorname{Re} i(z-z_0)^2 = t \operatorname{Im} (z-z_0)^2$$

For $t > 0$, the following <u>signature table</u> for $\operatorname{Re} i\theta$ is critical

(212.2)

<0	>0	$\operatorname{Re} i\theta$
>0	z_0 <0	

So we would like to move $e^{i\theta}$ into either the 2nd or 4th quadrants, and $e^{-i\theta}$ into the 1st or 3rd quadrants. As in the case of Szegö's Strong Limit Theorem, we must first separate $e^{i\theta}$ and $e^{-i\theta}$ algebraically. Based on our previous experience we should consider upper/lower or lower/upper factorizations of v_θ: we have.

$$v_\theta = \begin{pmatrix} 1 & re^{i\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar{r}e^{-i\theta} & 1 \end{pmatrix} \qquad \text{upper/lower}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{-\bar{r}e^{-i\theta}}{1-|r|^2} & 1 \end{pmatrix} \begin{pmatrix} 1-|r|^2 & 0 \\ 0 & \frac{1}{1-|r|^2} \end{pmatrix} \begin{pmatrix} 1 & re^{i\theta}/1-|r|^2 \\ 0 & 1 \end{pmatrix}$$

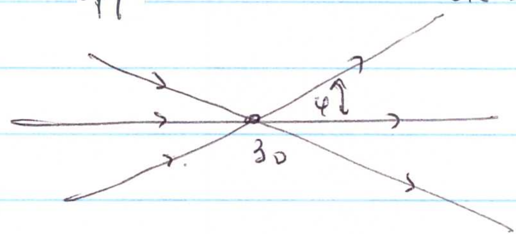
lower/upper.

For $z > z_0$, we use the upp/low fact, & for $z < z_0$, we use the lower/upp.

L/U $\quad \bullet \quad$ U/L

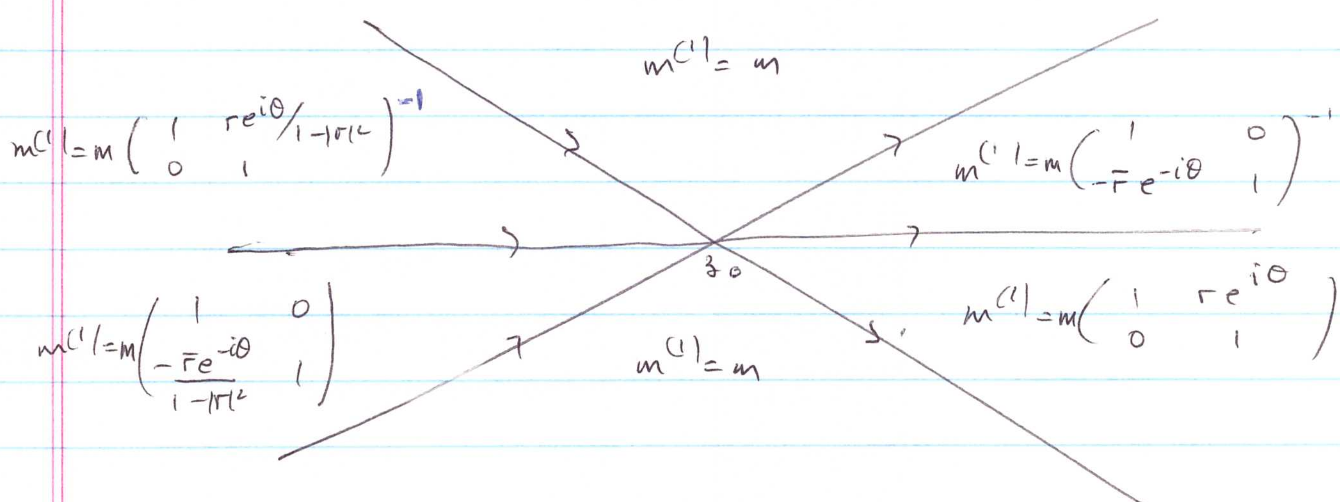
z_0

Now imagine for the moment that the functions $r(z)$, $\overline{r(z)}$, $r/_{1-|r|^2}$ and $-\bar{r}/_{1-|r|^2}$ have anal. continuations off the real axis. Extend

$\Sigma \to \Sigma^{(1)} =$ 

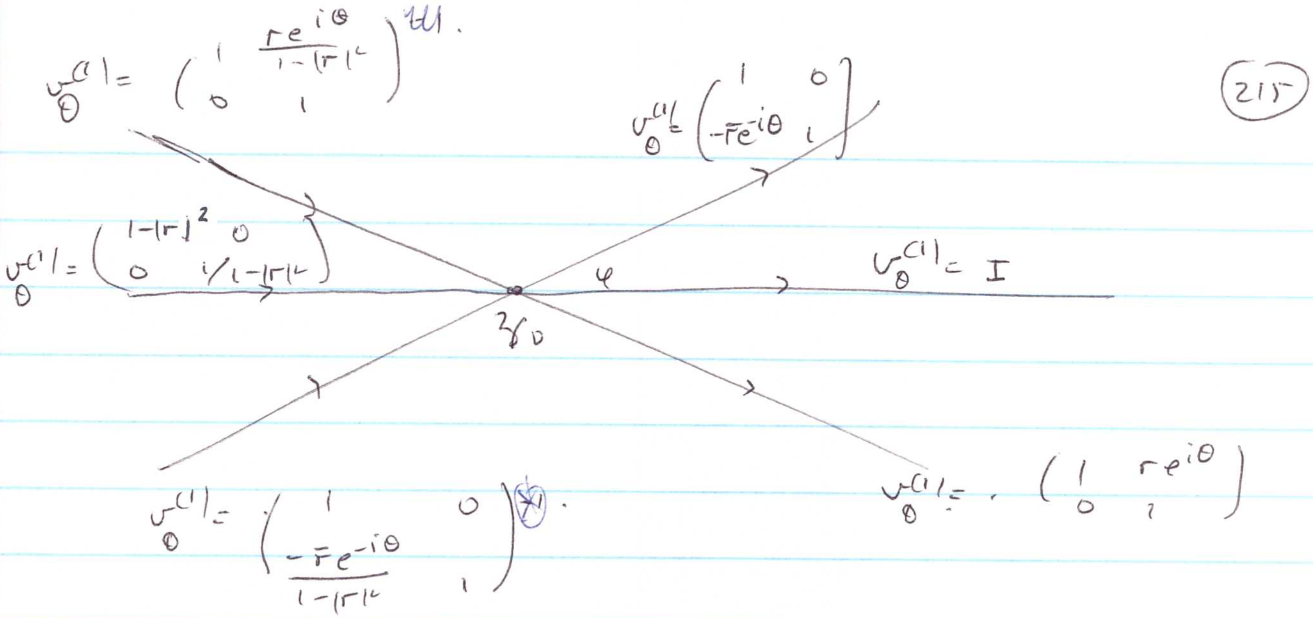
for some opening angle $\varphi > 0$. Deform the RHP $\Sigma \to \Sigma^{(1)}$ in the following way:

Define $m^{(1)}(z)$ via

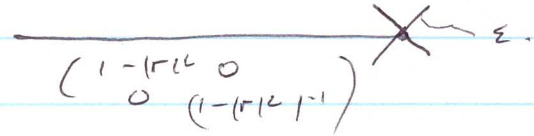


Then $m^{(1)}$ solves a normalized RHP $(\Sigma^{(1)}, v_\theta^{(1)})$

where



We see immediately clear that exponential factors are in the right quadrants of the signature table, eg on $\mathbb{R}_+ e^{i\varphi}$, $|e^{-i\theta}| = e^{-\operatorname{Re} i\theta} = e^{-t\,\operatorname{Im}(z-z_0)^2}$, which goes to zero exponentially as $t \to \infty$. Similarly for the left quadrants. Clearly the RHP localizes as $t \to \infty$ to a nbhood of $z = z_0$, apart from the ray $\{z < z_0\}$



The jump across $\{z < z_0\}$ can be removed in the following way: let $\delta_\pm \in 1 + \partial C(L^2)$ solve

The (scalar) normalized RHP

$$\delta_+ = \delta_-(1-|r|^2), \quad z<z_0$$
$$= \delta_-, \quad z>z_0$$

Such RHP's can of course be solved by formula

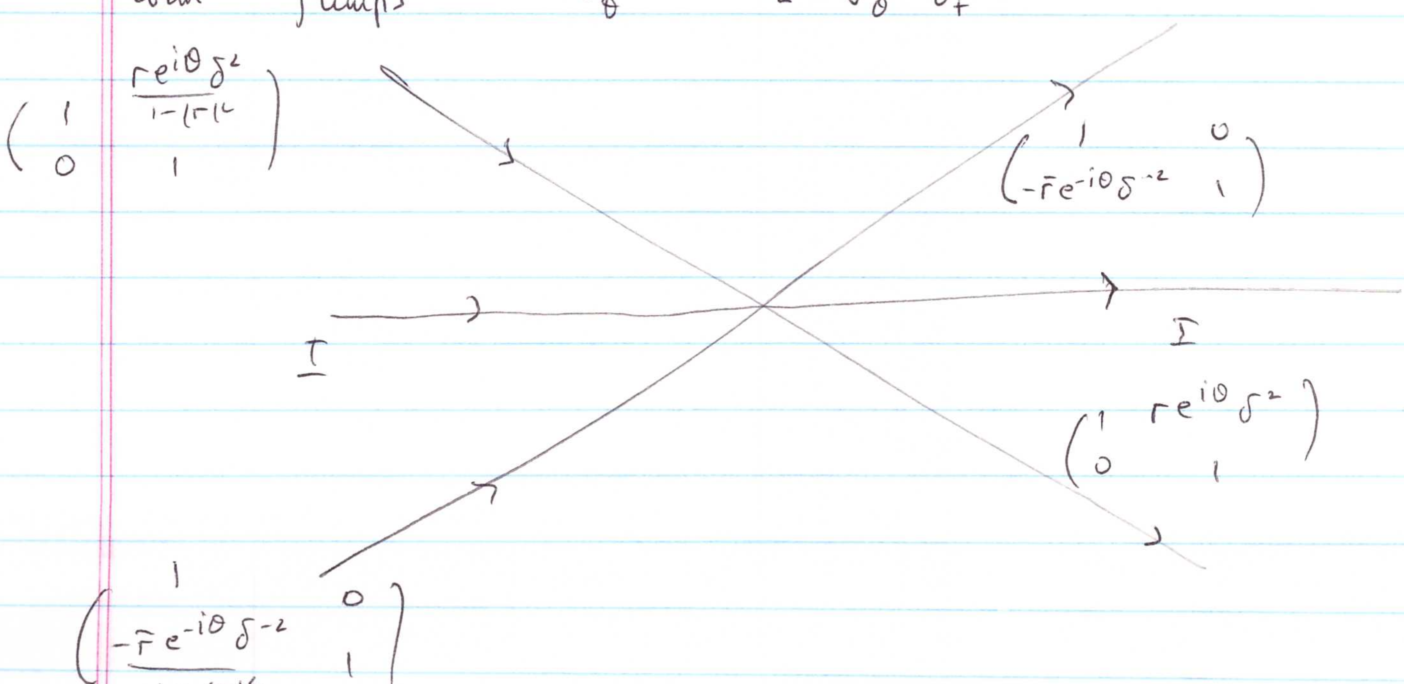
$$\delta(z) = e^{\frac{1}{2\pi i}\int_{-\infty}^{z_0} \frac{\log(1-|r(s)|^2)}{s-z}\,ds}.$$

Set

$$\tilde{m}^{(1)}(z) = m^{(1)}\begin{pmatrix} \delta(z)^{-1} & 0 \\ 0 & \delta(z) \end{pmatrix} = m^{(1)}\delta^{-\sigma_3}$$

Then $\tilde{m}^{(1)}(z)$ solves a normalized RHP on $\Sigma^{(1)}$

with jumps $\tilde{v}_\theta^{(1)} = \delta_-^{\sigma_3} v_\theta^{(1)} \delta_+^{-\sigma_3}$



where we have used the fact that

$$\delta_-^{\sigma_3}\begin{pmatrix} 1-|r|^2 & 0 \\ 0 & (1-|r|^2)^{-1}\end{pmatrix}\delta_+^{-\sigma_3}$$

$$= \begin{pmatrix} \frac{\delta_-(1-|r|^2)}{\delta_+} & 0 \\ 0 & \frac{\delta_+}{\delta_-(1-|r|^2)}\end{pmatrix} = I.$$

Now as $t \to \infty$ the RHP $(\Sigma^{(1)}, \tilde{v}_\theta^{(1)})$ fully localizes to an ε-nbhood of $z = z_0$

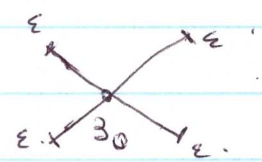


Fig 217.1

Note that although $\tilde{v}_\theta^{(1)} \to I$ pointwise on $\Sigma^{(1)}$, the convergence is not uniform: it becomes slower and slower as $z \to z_0$, $\operatorname{Re} i\theta = -t \operatorname{Im}(z-z_0)^2$.

One now factors $\delta = \delta_0\,\delta_1$ where

(217.1) $$\delta_0 = e^{\beta + i\nu(z_0)}\,(z-z_0)^{i\nu(z_0)}, \quad \delta_1 = e^{\xi(z,z_0)}$$

where $\nu(z_0) = -\frac{1}{2\pi}\log(1-|r(z_0)|^2)$ as above,

β is a constant given explicitly given in terms of $r(z)$, and $\xi(z, z_0)$

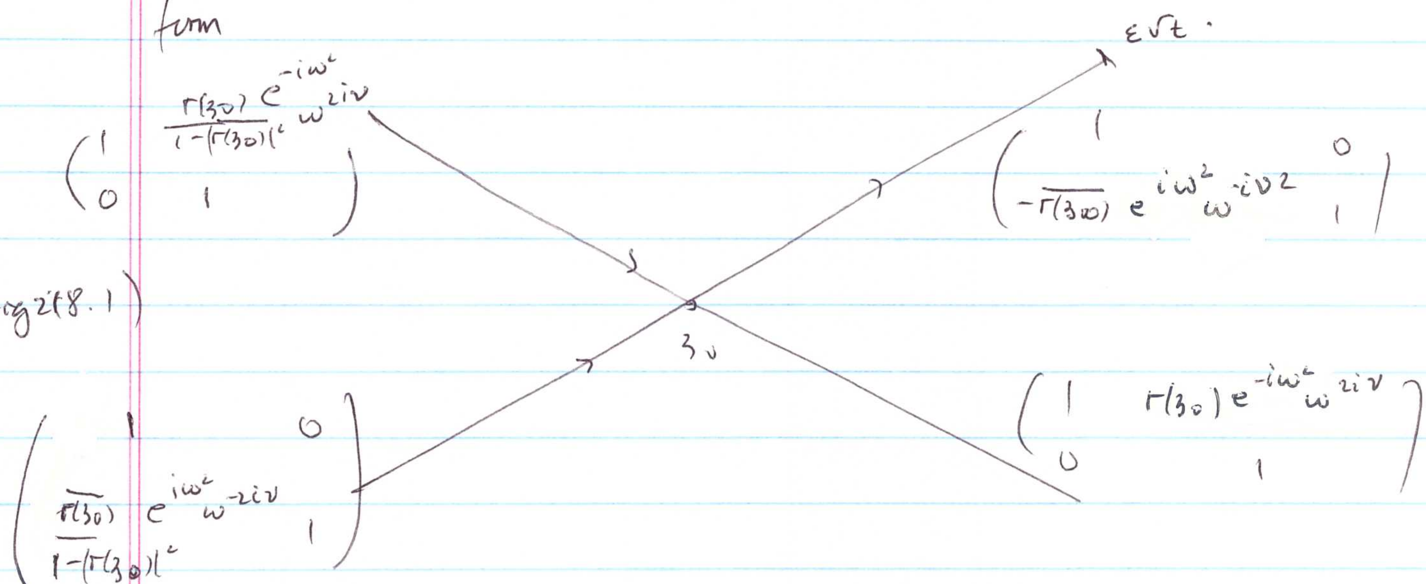
$\to 0$ at some rate as $z \to z_0$. Rescaling

$$\sqrt{t}(z-z_0) \to w$$

and conjugating out terms independent of w, we obtain

a RHP on a large cross of size $\varepsilon\sqrt{t}$ of the

form



(Fig 218.1)

Note that the rescaling $\sqrt{t}(z-z_0) \to w$ gives

rise to the logarithmic phase shift $e^{-i\nu(z_0)\log 2t}$ in (211.2)

via the term $(z-z_0)^{i\nu(z_0)} = w^{i\nu(z_0)}/t^{i\nu(z_0)} = w^{i\nu(z_0)} e^{-i\nu(z_0)\log t}$

Folding the RHP in (218.1) back to the real axis, ie reversing the step that led from $m \to m^{(1)}$, we obtain a normalized RHP on $\Sigma = \mathbb{R}$ with

$$v_\theta^{\#} = \begin{pmatrix} 1-|r(z_0)|^2 & r(z_0)e^{-i\omega^2} \\ -\bar{r}(z_0)e^{i\omega^2} & 1 \end{pmatrix}$$

$$= e^{-i\frac{\omega^2}{2}\sigma_3}\, v(z_0)\, e^{i\frac{\omega^2}{2}\sigma_3}.$$

To solve this RHP

$$m_+^{\#} = m_-^{\#}\, e^{-i\frac{\omega^2}{2}\sigma_3}\, v(z_0)\, e^{i\omega^2/2\,\sigma_3}.$$

we note that $m_\pm^{\#\#} = m_-^{\#} e^{-i\omega^2/2\,\sigma_3}$ solves a RHP with a jump matrix <u>indep. of ω</u> : $m_+^{\#\#} = m_-^{\#\#} v(z_0)$

Hence by the <u>mantra</u>, $m^{\#\#}$ solves an ode. This ode turns out to be solvable in terms of the classical parabolic cylinder functions $D_a(z)$. Identifying parameters then leads to the solution (211.1) (211.2).

All the technical difficulties in the above proof then boil down to showing that although $r, \bar{r}, \ldots$ may not be analytic, they can be approximated to high enough order by analytic (in fact, rational) functions.

(2) We now consider op's with weights of of the form $e^{-V(x)}dx$ on $\mathbb{R}$, $V(x) > 0$.

We always assume that all the moments are finite

$$\int |x|^m e^{-V(x)} dx < \infty \quad , m = 0, 1, 2, \ldots$$

Remark: Note the essential difference between the multipliers in the Szegö Str. Limit Problem and the NLS problem viz z^n vs $e^{i\theta}$. In the former case one factorization (lower/upper) is sufficient to yield a deformation with $|z|^n < 1$ in the jump matrix: but for $e^{i\theta}$, we need low/upp & upp/lower to achieve $|e^{i\theta}| < 1$, because of the signature table (212.2).

Recall from lecture 2 that the OP's wrt $e^{-V}dx$

$$p_n(x) = \gamma_n x^k + \cdots, \quad \gamma_n > 0, \quad n = 0, 1, \ldots,$$

$$\int p_n p_m e^{-V(x)} dx = \delta_{n,m} \qquad n, m \geq 0$$

can be expressed in terms of the following RHP:
(Fokas-Its-Kitaev)

Fix $n \geq 0$. Suppose

- $Y = Y^{(n)}(z)$ is anal. in $\mathbb{C} \setminus \mathbb{R}$
- $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & e^{-V(z)} \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R}$
- $Y(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \to I$

Then

(221.1) $$Y_{11}(z) = \pi_n(z) \equiv \frac{p_n(z)}{\gamma_n} = z^n + \cdots$$

$$= \text{monic orthog. poly wrt } e^{-V}dx.$$

Ques: How does π_n behave as $n \to \infty$?

By the above, the quest. becomes:
how does $Y = Y^{(n)}$ behave as $n \to \infty$.

The interesting situation is when V itself varies with n; here we consider

$$(222.1) \qquad V(z) = n\,Q(z) \qquad , \quad Q(z) > 0 .$$

In the previous 2 asymptotic problems, we could "see" where the leading term was coming from eg for NLS, we were led to consider a nbhood of the stationary phase point z_0. But it is far from clear where the main contribution to $Y = Y^{(n)}(z; e^{-nV})$ is coming from.

We proceed by a sequence of transformations (deformations): (see D-Kriecherbauer-McLaughlin-Venakides-Zhou, I, II, CPAM, 1999)

Step 1 $\quad Y \to T$.

Let $g(z)$ be a scalar function with the following properties

- $g(z)$ is anal. in $\mathbb{C} \setminus \mathbb{R}$
- $g(z) = \log z + o(1)$ as $z \to \infty$ in $\mathbb{C} \setminus \mathbb{R}$.

Let ℓ be a constant, to be determined.
Set

$$T(z) = e^{-n\frac{\ell}{2}\sigma_3}\, Y\, e^{-n(g(z) - \frac{\ell}{2})\sigma_3}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $T(z)$ solves the following <u>normalized</u> RHP on $\mathbb{R}$:

- $T(z)$ is anal. in $\mathbb{C} \setminus \mathbb{R}$
- $$T_+(z) = e^{-n\frac{\ell}{2}\sigma_3}\, Y_+\, e^{-n(g_+ - \frac{\ell}{2})\sigma_3} = e^{-n\frac{\ell}{2}\sigma_3}\, Y_- \begin{pmatrix} 1 & e^{-nV} \\ 0 & 1 \end{pmatrix} e^{-n(g_+ - \frac{\ell}{2})\sigma_3}$$
$$= T_-\, e^{n(g_- - \frac{\ell}{2})\sigma_3} \begin{pmatrix} 1 & e^{-nV} \\ 0 & 1 \end{pmatrix} e^{-n(g_+ - \frac{\ell}{2})\sigma_3}.$$

ie

$$T_+ = T_-\, v_T \quad \text{on } \mathbb{R}$$

where
$$v_T = \begin{pmatrix} e^{n(g_+ - g_-)} & e^{-n(V - g_+ - g_- - \ell)} \\ 0 & e^{n(g_- - g_+)} \end{pmatrix}$$

- $$T(z) = e^{-n\ell/2\,\sigma_3}\, Y\, e^{-n(\log z - \frac{\ell}{2})\sigma_3}\, (1 + o(1))$$
$$\longrightarrow I \quad \text{as } z \to \infty$$

Now suppose in addition that g satisfie the following further properties: there $\exists$ a closed finite interval $I \subset \mathbb{R}$ st

(224.1) $$g_+(s) + g_-(s) - V(s) - \ell = 0 \qquad \text{for } s \in I$$

(224.2) $$g_+(s) - g_-(s) \text{ is purely imaginary} \qquad \text{for } s \in I$$
$$\text{and } \quad i\frac{\partial}{\partial s}(g_+ - g_-) > 0 \qquad \text{for } s \in \mathrm{int}(I)$$

(224.3) $$g_+(s) + g_-(s) - V(s) - \ell < 0 \qquad \text{for } s \in \mathbb{R} \setminus I$$

(224.4) $$e^{g_+(s) - g_-(s)} = 1 \qquad \text{for } s \in \mathbb{R} \setminus I$$

<u>Step 3</u> $T \to S$

The significance of the conditions (224.1)–(224.2) is the following: inserting these conditions into v_T, we obtain

$$v_T = \begin{pmatrix} 1 & e^{-n(V - g_+ - g_- - \ell)} \\ 0 & 1 \end{pmatrix} \qquad I \qquad v_T = \begin{pmatrix} 1 & e^{-n(V - g_+ - g_- - \ell)} \\ 0 & 1 \end{pmatrix}$$

$$v_T = \begin{pmatrix} e^{n(g_+ - g_-)} & 1 \\ 0 & e^{-n(g_+ - g_-)} \end{pmatrix}$$

Thus for $z \in \mathbb{R} \setminus I$, $v_T \to I$ as $n \to \infty$ by (224.3). On I, setting $G = g_+ - g_-$

(225.0) $$v_T = \begin{pmatrix} 1 & 0 \\ e^{nG} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-nG} & 1 \end{pmatrix} \equiv v_- v_0 v_+ .$$

Now observe that on I, by (224.1)

$$G = g_+ - g_- = 2g_+ - V - \ell$$

$$= -2g_- + V + \ell$$

Hence if

(225.1) $V(z)$ is anal. in a nbhood of $\mathbb{R}$

we see that G has an analytic continuation to $\mathbb{C}^+$ and $\mathbb{C}^-$ ie G is anal. in a nbhood of I.

Furthermore by (224.2)

$$\frac{\partial}{\partial s} i(g_+ - g_-) = \frac{\partial}{\partial s} - \mathrm{Im}\, G > 0$$

But if y denotes a transverse variable on I ,

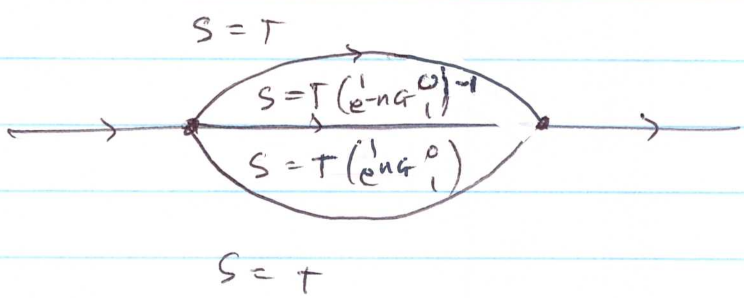
then by Cauchy

$$\frac{\partial \,\mathrm{Re}\, G}{\partial y} = -\frac{\partial}{\partial x} \mathrm{Im}\, G > 0 .$$

Thus as $\operatorname{Re} G(z) = 0$ on I, we must have

(226.1)
$$\begin{cases} \operatorname{Re} G(z) > 0 & \text{for } \operatorname{Im} z > 0 \\ \operatorname{Re} G(z) < 0 & \text{for } \operatorname{Im} z < 0 . \end{cases}$$

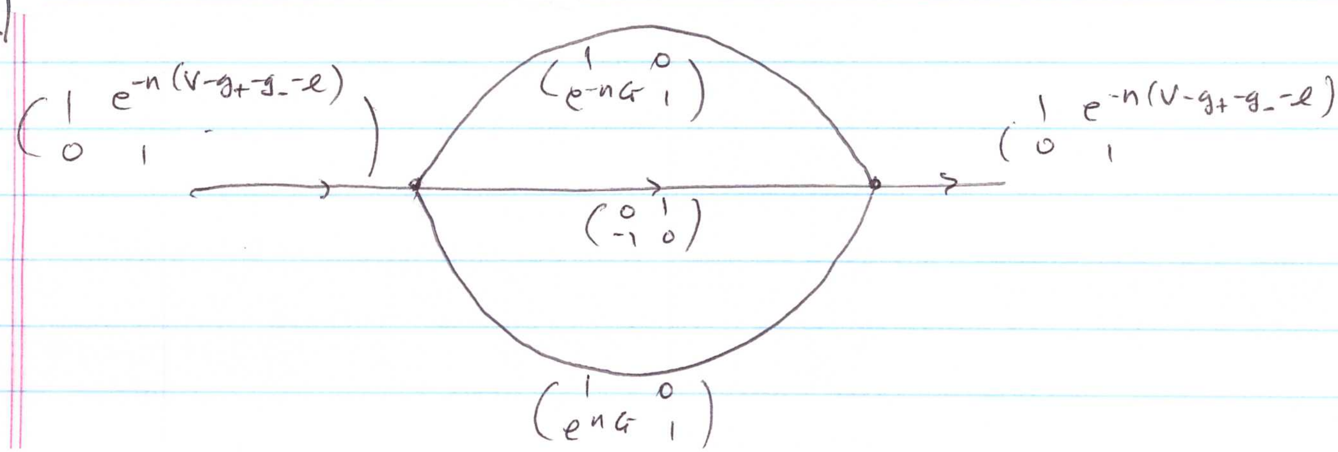
We now extend $\mathbb{R}$ to a lens-shaped region Σ_s, and bearing (225.0) in mind we define $S(z)$ as a piecewise analytic function as follows



Σ_s

Then S solves a normalized RHP on Σ_s with jump matrix v_S given by

$\boxed{v_S}$



Clearly

$$v_S \to v^\infty$$

pointwise as $n \to \infty$, where

$$v^\infty = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{on } I$$

$$= I \quad \text{elsewhere on } \Sigma_S.$$

Thus we expect S to converge to S^∞ as $n \to \infty$, where S^∞ is the solution of the normalized RHP

- $S^\infty(z)$ anal. in $\mathbb{C} \setminus I$
- $S^\infty_+ = S^\infty_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on I
- $S^\infty(z) \to I$ as $z \to \infty$.

This RHP is a direct sum of scalar problems and hence easily solvable by formula. Thus we expect for $z \in I$, say

$$Y_+ = e^{n\ell/2\,\sigma_3}\, T_+\, e^{n(g_+ - \ell/2)\sigma_3}$$

$$= e^{n\ell/2\,\sigma_3}\, S_+ \begin{pmatrix} 1 & 0 \\ e^{-nG_+} & 1 \end{pmatrix} e^{n(g_+ - \frac{\ell}{2})\sigma_3}$$

$$\sim e^{n\ell/2\,\sigma_3}\, S^\infty \begin{pmatrix} 1 & 0 \\ e^{-nG_+} & 1 \end{pmatrix} e^{n(g_+ - \ell/2)\sigma_3}$$

as $n \to \infty$, yielding, in particular the asymptotics of $\pi_n(z) = Y_{11}(z)$.

<u>Now some facts:</u>

(i) the above analysis goes through for any $V(z)$ anal. in a nbhood of $\mathbb{R}$ st

$$\frac{V(z)}{\log|z|} \to \infty \quad \text{as } |z| \to \infty$$

(ii) For such V, a "g-function" with prop's (224.1) – (224.4) can be shown to exist, but instead of 1 interval I, there can be a finite # of intervals $\overline{\;I_1\;}\ \overline{\;I_2\;}\ \cdots\ \overline{\;I_k\;}$

The g-function is given in terms of the <u>equilibrium measure</u> for the OP's, $g(z) = \int \log(z-s)\, d\mu_{eq}(s)$

- Although $v_S \to \begin{pmatrix}0&1\\-1&0\end{pmatrix}$ ptwise as $n\to\infty$, the convergence is not uniform near the end points of I, or, more generally, the end points of $\bigcup_{j=1}^{k} I_j$ ... One controls the situation by constructing certain explicit parametrices for the problem near these points

- the solution of the limiting problem S^∞

$\boxed{v_S,\ n\to\infty}$

(**)

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix} \quad \begin{pmatrix}e^{in\Omega_1}&0\\0&e^{-in\Omega_1}\end{pmatrix} \quad \begin{pmatrix}0&1\\-1&0\end{pmatrix} \quad \begin{pmatrix}e^{in\Omega_2}&0\\0&e^{-in\Omega_2}\end{pmatrix} \quad \cdots \quad \begin{pmatrix}e^{in\Omega_{k-1}}&0\\0&e^{-in\Omega_{k-1}}\end{pmatrix} \quad \begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

can be expressed explicitly in terms of the theta functions on the Riemann surface naturally assoc. with (2 copies of) $\mathbb{C}\setminus\left(\bigcup_1^k I_j\right)$.

(For $k>2$, and z between the intervals I_j, $v_S(z)$ oscillates in an elementary way as $n\to\infty$: see (**) below).