Lecture 2 The RHP's for MKdV and Painlevé II were presented out of the blue and without a derivation or a proof. Here is an example (see P.D OP's and Random Matrices) that one can verify directly.

Let $d\mu(x) = w(x)dx$, $w(x) \geq 0$, be a measure with finite moments i.e.

$$\int_{\mathbb{R}} |x|^k w(x) dx < \infty \qquad k=0,1,2,\ldots$$

and let $(p_k(x))_{k\geq 0}$ be the orthonormal polynomials associated with $d\mu(x)$ obtained by orthogonalizing $1, x, x^2, \ldots$ wrt $d\mu(x)$ (Gram-Schmidt procedure). We have

$$p_k(x) = \gamma_k x^k + \ldots, \quad \gamma_k > 0, \quad k \geq 0,$$

and

(19.1) $$\int p_k(x) p_j(x) w(x) dx = \delta_{jk}, \qquad j,k \geq 0$$

Exercise The p_k's $\exists$ and are unique, $k \geq 0$

Set $\pi_k(x) = \frac{1}{\gamma_k} p_k(x) = x^k + \ldots$; π_k is the monic orthog.

poly. wrt $d\mu$.

Orthogonal polynomials (OP's) are fundamental objects in analysis and their asymptotics as $k \to \infty$ are of great interest. For example, the proof of universality for invariant random matrix ensembles, reduces to the evaluation of the asymptotics of OP's. Some of the classical OP's are the following:

- $d\mu(x) = e^{-x^2} dx \Rightarrow p_n(x) = c_n H_n(x) = c_n \times$ Hermite poly, c_n const.

- $d\mu(x) = (1-x)^\alpha (1+x)^\beta dx$, $-1 < x < 1$, $\alpha, \beta > -1$ $\Rightarrow$ Jacobi polynomials

- $d\mu(x) = e^{-x} dx$, $x > 0$ $\Rightarrow$ Laguerre polynomials.

Each of the classical poly's has an integral representation eg (see Szegö "Orthog. Polynomials")

$$(20.1) \quad H_n(x) = \frac{n!}{2\pi i} \int_{\mathcal{C}} w^{-n-1} e^{2xw - w^2} dw, \quad \mathcal{C} \text{ encloses } 0.$$

As noted before, the asymptotics of $H_n(x)$ as $n \to \infty$ (and also as $|x| \to \infty$) can then be inferred from (20.1) using the classical steepest descent method. However for $e^{-x^4}dx$, say, no such integral formulae are known. <u>However, we do have a RHP!</u>

Let $w(x)\,dx$ have finite moments as above. Let $\Sigma = \mathbb{R}$ oriented from $-\infty \to +\infty$ and let $v(z) = \begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix}$, $z \in \mathbb{R}$. Fix $n \geq 0$. We seek a 2×2 matrix-valued function $Y = Y^{(n)}(z)$ such that

$$
(21.1)\quad \begin{cases}
\bullet\ Y = Y^{(n)}(z) \text{ is analytic in } \mathbb{C} \setminus \Sigma \\
\bullet\ Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R}, \\
\bullet\ \text{where } Y_\pm(z) = \lim_{\varepsilon \downarrow 0} Y(z \pm i\varepsilon), \quad z \in \mathbb{R} \\
\bullet\ Y(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \to I \text{ as } z \to \infty.
\end{cases}
$$

(The proofs and calculations that follow are somewhat formal: fully rigorous proof later)

<u>Claim:</u> If Y exists, it is unique

Indeed if Y exists, then $\det Y(z)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$ and

$$(\det Y)_+(z) = (\det Y)_-(z) \det\begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix} = (\det Y)_-(z) \quad , z\in\mathbb{R}$$

Hence $\det Y$ is analytic across $\Sigma = \mathbb{R}$, and hence entire

But as $z\to\infty$ $\det Y(z) = \det\left(Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix}\right) \to 1$

and so $\det Y(z) \equiv 1$, by Liouville. Now suppose $\tilde{Y}(z)$ is another solution of (21.1): set $R(z) = \tilde{Y}(z)\, Y^{-1}(z)$.

Note that as $\det Y(z) \equiv 1$, $Y^{-1}(z)$ $\exists$ and is analy. in $\mathbb{C}\setminus\mathbb{R}$. Thus

- $R(z)$ is anal. in $\mathbb{C}\setminus\mathbb{R}$

- $$R_+(z) = \tilde{Y}_+(z)\,(Y_+(z))^{-1} = \left(\tilde{Y}_-(z)\begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix}\right)\left(Y_-(z)\begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix}\right)^{-1} = R_-(z) \quad , z\in\mathbb{R}.$$

and so $R(z)$ is analytic across Σ and hence entire.

$$\cdot \; R(z) = \tilde{Y}(z)\,(Y(z))^{-1} = \left[\tilde{Y}(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^{n}\end{pmatrix}\right]\left(Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^{n}\end{pmatrix}\right)^{-1}$$

$$\to I \quad \text{as } n \to \infty$$

Hence $R(z) \equiv I$, again by Liouville $\Rightarrow \tilde{Y}(z) = Y(z)$.

Remark: This proof, simple as it is, is prototypical in RHP th^y.

Write

(23.1).
$$Y(z) = \begin{pmatrix} Y_{11}(z) & Y_{12}(z) \\ Y_{21}(z) & Y_{22}(z)\end{pmatrix} = \left(I + O\left(\tfrac{1}{z}\right)\right)\begin{pmatrix} z^{n} & 0 \\ 0 & z^{-n}\end{pmatrix}.$$

Suppose $n \geq 1$. The first row of $Y_+ = Y_- \begin{pmatrix} 1 & w \\ 0 & 1\end{pmatrix}$

reads as follows

(23.2)
$$(Y_{11} \;\; Y_{12})_+ = (Y_{11} \;\; Y_{12})_- \begin{pmatrix} 1 & w \\ 0 & 1\end{pmatrix}$$

In particular

$$(Y_{11})_+ = (Y_{11})_-$$

$\Rightarrow$ $Y_{11}(z)$ is entire. But from (23.1)

$$Y_{11}(z) = z^n\left(1 + O\left(\tfrac{1}{z}\right)\right) \quad \text{as } z \to \infty$$

Hence
$$Y_{11}(z) = \text{polynomial} = z^n + \cdots.$$

Now

$$(Y_{12})_+ = (Y_{12})_- + w\, Y_{11}(z).$$

Set

(24.1) $$h(z) \equiv \int_{\mathbb{R}} \frac{w(x)\, Y_{11}(x)}{x-z} \frac{dx}{2\pi i}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Note that as $w(x)dx$ has finite moments, this integral $\exists$. Then $h(z)$ is analy. in $\mathbb{C}\setminus\mathbb{R}$ and for $z \in \mathbb{R}$,

$$h_+(z) - h_-(z) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \left(\frac{1}{x-z-i\varepsilon} - \frac{1}{x-z+i\varepsilon} \right) w\, Y_{11} \frac{dx}{2\pi i}$$

$$= \lim_{\varepsilon \downarrow 0} \int \frac{2i\varepsilon}{(x-z)^2 + \varepsilon^2}\, w(x)\, Y_{11}(x) \frac{dx}{2\pi i}$$

$$\overset{\uparrow}{=} w(z)\, Y_{11}(z).$$

exercise!

Thus

$$(Y_{12} - h)_+ = (Y_{12} - h)_-$$

and so $Y_{12} - h$ is entire. But from (23.1)

$$Y_{12}(z) = O(z^{-n-1}) \quad \text{as } z \to \infty \quad \text{and} \quad h(z) = O\left(\tfrac{1}{z}\right)$$

as $z \to \infty$; hence $(Y_{12} - h)(z) \to 0$ as $z \to \infty$. Thus, again

(24.2) by Liouville, $$Y_{12}(z) = h(z) = \int_{\mathbb{R}} \frac{w(x)\, Y_{11}(x)}{x-z} \frac{dx}{2\pi i}.$$

Expanding (24.2) as $z \to \infty$, we have

(25.1) $$Y_{12}(z) = \frac{-1}{2\pi i} \int w(x)\, Y_{11}(x) \left(\frac{1}{z} + \frac{x}{z^2} + \dots + \frac{x^{n-1}}{z^n} + \frac{x^n}{z^{n+1}} + \dots \right) dx$$

As $Y_{12}(z) = O(z^{-n-1})$ as $n \to \infty$, we must have

(25.2) $$\int Y_{11}(x)\, x^j\, w(x)\, dx = 0, \quad 0 \le j \le n-1.$$

It follows that necessarily

$$Y_{11}(x) = \pi_n(x) = n^{th} \text{ monic OP for } w(x)\,dx.$$

Thus

(25.3) $$\left(Y_{11}(z) \;\; Y_{12}(z) \right) = \left(\pi_n(z) \;\; \int_{\mathbb{R}} \frac{\pi_n(x)\, w(x)}{x-z} \frac{dx}{2\pi i} \right)$$

Similarly (<u>exercise</u>) we find that

(25.4) $$\left(Y_{21}(z) \;\; Y_{22}(z) \right) = \left(-2\pi i\, \gamma_{n-1}^2\, \pi_{n-1}(z) \;\; -2\pi i\, \gamma_{n-1}^2 \int \frac{\pi_{n-1}\, w}{x-z} \frac{dx}{2\pi i} \right)$$

Conversely, if $\{Y_{ij}\}_{1 \le i,j \le 2}$ are given as in (25.3)(25.4), then $Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$ satisfies (21.1). We have proved the following result:

Proposition 26.1 (Fokas – Its – Kitaev). For $n \geq 0$,

$$Y(z) = \begin{pmatrix} \pi_n(z) & \int_{\mathbb{R}} \frac{w(x)\pi_n(x)}{x-z}\frac{dx}{2\pi i} \\ -2\pi i \gamma_{n-1}^2 \pi_{n-1}(x) & \int_{\mathbb{R}} \frac{-2\pi i \gamma_{n-1}^2 \pi_{n-1}(x)\, w(x)}{x-z}\frac{dx}{2\pi i} \end{pmatrix}$$

is the unique solution of the RHP $\left(\Sigma = \mathbb{R},\ v = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\right)$

normalized so that

(26.2) $\qquad \circ \quad Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \to I$ as $z \to \infty$.

(For $n=0$, $Y = \begin{pmatrix} 1 & \int_{\mathbb{R}} \frac{w(x)}{x-z}\frac{dx}{2\pi i} \\ 0 & 1 \end{pmatrix}$)

Applying the non-linear steepest descent method to (Σ, v) as $n \to \infty$, we can, in particular deduce the asymptotics of $\pi_n(z)$, for very general weights $w(x)\,dx$.

Orthogonal polynomials famously satisfy a 3-term recurrence relation

(26.3) $$b_{n-1} p_{n-1}(x) + a_n p_n(x) + b_n p_{n+1}(x) = x\, p_n(x), \quad n \geq 0,$$

$a_n \in \mathbb{R}$, $b_n > 0$ for $n \geq 0$ ($b_{-1} \equiv 0$). As $n \to \infty$, one is interested not only in the asymptotics of $\pi_n(x)$ but also in the asymptotics of a_n, b_n and the norming constants γ_n. These 3 quantities can be read off directly from the RHP: we have (exercise)

(27.1)
$$\begin{cases} \gamma_{n-1}^2 = -\frac{1}{2\pi i} \left(Y_1^{(n)}\right)_{21} \\ a_n = \left(Y_1^{(n)}\right)_{11} - \left(Y_1^{(n+1)}\right)_{11} \\ b_{n-1}^2 = \left(Y_1^{(n)}\right)_{12} \left(Y_1^{(n)}\right)_{21} \end{cases}$$

where
$$Y = Y^{(n)} = \left(I + \frac{1}{z} Y_1^{(n)} + O\left(\frac{1}{z^2}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \to \infty.$$

So the RHP captures all the quantities of basic interest for OP's.

Up till now, we have presented RHP's as a tool to evaluate physical and mathematical quantities asymptotically as some associated parameter goes to infinity. But RHP's are also very useful in analytical and algebraic contexts. We will develop these themes later, but just to illustrate how things work, we will now show how the RHP for the OP's can be used to derive the difference equation (26.3).

There is the following basic mantra for RHP's: if the jump matrix v of a RHP (Σ, v) is <u>independent</u> of a parameter in the problem, then variations wrt that parameter give rise to differential / difference equations.

To apply this mantra to the RHP for OP's, observe that n only appears in the asymptotics $Y\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \to I$, but not in $v = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$.

Let $Y^{(n)}, Y^{(n+1)}$ be the solutions of the OP RHP (21.1). Let $T = Y^{(n+1)} (Y^{(n)})^{-1}$. Then

- $T(z)$ is analy. in $\mathbb{C} \setminus \mathbb{R}$.

- $$T_+(z) = Y_+^{(n+1)}(z) \left(Y_+^{(n)}(z) \right)^{-1}$$
$$= \left(Y_-^{(n+1)}(z)\, v(z) \right) \left(Y_-^{(n)}(z)\, v(z) \right)^{-1}$$
$$= T_-(z) \qquad , z \in \mathbb{R}.$$
and so $T(z)$ is entire, as before.

- As $z \to \infty$
$$T(z) = Y^{(n+1)}(z) \left(Y^{(n)}(z) \right)^{-1}$$
$$= Y^{(n+1)}(z) \begin{pmatrix} z^{-n-1} & 0 \\ 0 & z^{n+1} \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \left[Y^{(n)}(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \right]^{-1}$$
$$= z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + O(1) \quad \text{as } z \to \infty.$$

Thus by Liouville
$$T(z) = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} + A_n \quad \text{for some constant matrix } A_n.$$

In other words, we have shown that

$$Y^{(n+1)}(z) = \left(\begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} + A_n\right) Y^{(n)}(z)$$

for some constant matrix A_n. Evaluating A_n, we then obtain, in particular, the recurrence relation (26.3) (exercise).

Now from an analytic point of view, what kind of problem is a RHP? As we will see the Cauchy operator (cf (24.11))

$$Ch(z) = \int_\Sigma \frac{h(s)}{s-z} \frac{ds}{2\pi i}, \qquad z \in \mathbb{C} \setminus \Sigma$$

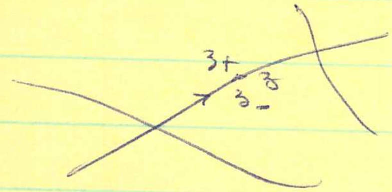
will play a central role. We saw above that for $\Sigma = \mathbb{R}$,

$$C^+h(z) - (C^-h)(z) = h(z) \qquad z \in \mathbb{R}$$

i.e.

(30.1) $$C^+ - C^- = 1 = id$$

The same is true for the Cauchy operator on very general oriented contours Σ



Let a jump matrix v on Σ be given and suppose $m(z)$ solves the normalized RHP (Σ, v). Then

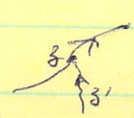
- $m(z)$ is analytic in $\mathbb{C} \setminus \Sigma$
- $m_+(z) = m_-(z)\, v(z)$, $z \in \Sigma$
- $m(z) \to I$ as $z \to \infty$

How do we compute $m(z)$?

<u>Method:</u> Let $\mu(z) = I + f(z)$, $z \in \Sigma$, where $f(z) \to 0$ as $z \to \infty$ in Σ in some specific sense and suppose μ solves the singular integral equation

(31.1) $$(1 - C_v)\mu = I$$

Here I on the RHS denotes the matrix function which is constantly I on Σ and



$$C_v \mu(z) \equiv C^-(\mu(v-1))(z) = \lim_{z' \to z_-} (C(\mu(v-1)))(z')$$

Alternatively, writing $\mu = I + f$, we have

$$(1 - C_v)(I + f) = I$$

i.e.

(32.1) $$(1 - C_v) f = C_v I = C^-(v-1)$$

Now set

(32.2) $$m(z) = I + C(\mu(v - I))(z) \quad , \quad z \in \mathbb{C}\setminus\Sigma$$

$$= I + \int_\Sigma \frac{\mu(s)(v(s) - I)}{s - z} \frac{ds}{2\pi i}$$

Clearly • $m(z)$ is anal. in $\mathbb{C}\setminus\Sigma$, and • $m(z) \to I$ as $z \to \infty$.

Then on Σ

$$m_+ = I + C^+\mu(v - I)$$

$$= I + C^-(\mu(v - I)) + \mu(v - I) \quad , \text{ by } (30.)$$

$$= I + C_v\mu + \mu(v - I)$$

$$= \mu + \mu(v - I) \quad , \text{ by } (31.1)$$

$$= \mu v \; ,$$

Similarly

$$m_- = I + C^-\mu(v - I)$$

$$= \mu$$

Hence $m_+ = \mu v = m_- v$ and so $m(z)$ solves the normalized RHP (Σ, v).

In this way the RHP reduces to the study of the singular integral equation (31.1). Questions:

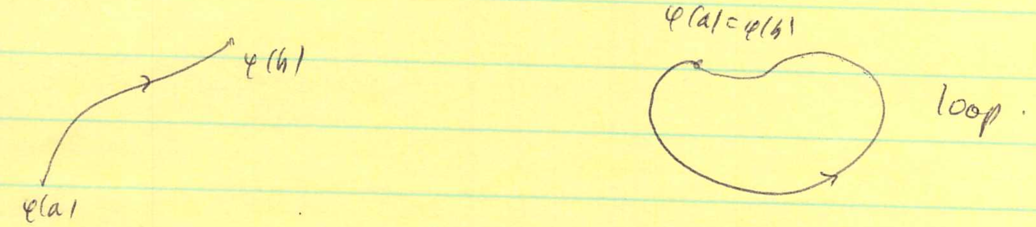
- In which space do we try to solve the equation? At the very least the operators $C^{\pm}$ should be bdd in this space.
- Is the solution unique in this space?
- Has the operator $1 - C_v$ any special structure? In particular, is it Fredholm?

The RHP is studied in a variety of spaces (see eg Clancey and Goh'berg). For example

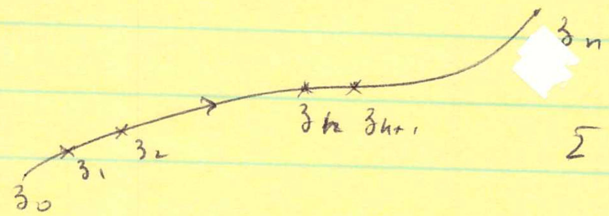
- $L^p(\Sigma, |dz|)$ spaces $1 < p < \infty$
- $W^{k,p}(\Sigma, |dz|)$ spaces $1 < p < \infty$, $k = 0, 1, 2, \ldots$
- spaces of Hölder continuous functions

Most commonly we study the RHP in L^p, $1 < p < \infty$; L^2 is the most important case. We will only consider the RHP in L^p, $1 < p < \infty$. Question: What conditions do we need to impose on Σ, to obtain a viable L^p Theory? Firstly we need a measure on Σ in order to create a measure space on Σ. Consider first the case where Σ is a simple continuous curve in $\mathbb{C}$ i.e.

$\Sigma = \{\varphi(t) : a \le t \le b\}$ where $\varphi(t)$ is continuous where $\varphi(t) = \varphi(t') \Rightarrow t = t'$ unless, in the case that Σ is a loop, and then $\varphi(a) = \varphi(b)$



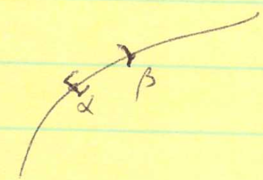
We allow for the possibility that $\varphi(a)$ or $\varphi(b) = \infty$.
The ordering $a \to b$ pushes forward to a natural orientation on Σ. The minimal, standard way to place a measure on Σ is to require that Σ be locally rectifiable, i.e. if z_0, z_n are 2 pts on Σ, and $z_0, z_1, \dots, z_n$ is any partition of $[z_0, z_n]$. (z_{i+1} succeeds z_i in the ordering on Σ, etc)



then

$$L = \sup_{\text{all partitions } \{z_i\}} \sum_{i=0}^{n-1} |z_i - z_{i+1}| < \infty$$

L is the <u>arc-length</u> of Σ from z_0 to z_n. For any interval $[\alpha, \beta)$ on Σ



we define

$$\mu([\alpha, \beta)) = \text{arc length } \alpha \to \beta$$

Now the sets $\{[\alpha, \beta) : \alpha < \beta \text{ on } \Sigma\}$ form a semi-algebra (see Royden: intersection of 2 such sets is again a set of the same type and the complement of $[\alpha, \beta)$ is a disjoint union of such sets) and hence μ can be extended to a complete measure on a sigma algebra $\mathcal{A}$ containing the Borel sets: the restriction of the