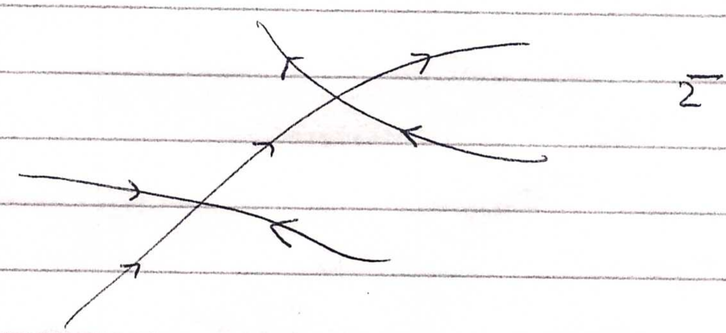
measure to the Borel sets is unique. We can now define $L^p(\Sigma, \mu) \equiv \{ f : f \text{ mble wrt } \mathcal{A} \text{ on } \Sigma, \int_\Sigma |f(z)|^p d\mu(z) < \infty \}$, $1 \le p < \infty$

and all the "usual" properties go through. One usually writes $\mu = |dz|$.

<u>Note</u>: <u>Exercise</u> $|dz|$ is also equal to Hausdorf-1 measure on Σ

<u>Lecture 3</u>

We will always assume the <u>Σ is a finite union of (simple, oriented) rectifiable curves which have only a finite # of points of intersection</u> eg



Note that if $\Sigma_1 = \mathbb{R}$ $\longrightarrow$

and $\Sigma_2 = \{ (x, x^3 \sin \frac{1}{x}) : x \in \mathbb{R} \}$

then $\Sigma = \Sigma_1 \cup \Sigma_2$ is not allowed, although Σ_1 & Σ_2

are both rectifiable.

For Σ as above we can define the Cauchy operator for $h \in L^p(\Sigma, |dz|)$, $1 \le p < \infty$, by

(37.1) $$Ch(z) = C_\Sigma h(z) \equiv \int_\Sigma \frac{h(\xi)}{\xi - z} \frac{d\xi}{2\pi i}, \quad z \in \mathbb{C} \setminus \Sigma$$

Here the integral is a line integral: if we parameterize Σ by the arc length s, $0 \le s \le s_0$, $\xi = \xi(s)$ then $\left|\frac{d\xi(s)}{ds}\right| = 1$ (why?) and (37.1) is given by

$$Ch(z) = \int_0^{s_0} \frac{h(\xi(s))}{\xi(s) - z} \frac{d\xi(s)}{ds} \frac{ds}{2\pi i}, \quad z \in \mathbb{C} \setminus \Sigma$$

The integrand (clearly) lies in $L^p(ds; [0, s_0])$.

We are interested in the boundary values

(37.2) $$C^\pm h(z) = C_\Sigma^\pm h(z) = \lim_{z' \to z\pm} Ch(z'), \quad z \in \Sigma$$

whenever these limits exist.

The limit in (37.2) can be decomposed in the following way: consider the case $\Sigma = \mathbb{R}$. For $z = x + i\varepsilon$, $\varepsilon > 0$, $x \in \mathbb{R}$, we have

(38.1) $$Ch(x+i\varepsilon) = \int_{\mathbb{R}} \frac{h(t)}{t - x - i\varepsilon} \frac{dt}{2\pi i}$$

$$= \int \frac{t - x + i\varepsilon}{(t-x)^2 + \varepsilon^2} h(t) \frac{dt}{2\pi i}$$

$$= \int \frac{1}{2\pi} \frac{\varepsilon}{(t-x)^2+\varepsilon^2} h(t)\, dt + \frac{1}{2\pi i} \int \frac{t-x}{(t-x)^2+\varepsilon^2} h(t)\, dt.$$

$$= \frac{1}{2} \int \frac{1}{\pi} \frac{1}{u^2+1} h(x+\varepsilon u)\, du$$

$$+ \frac{1}{2\pi i} \int_{|t-x|<\varepsilon} \frac{t-x}{(t-x)^2+\varepsilon^2} h(t)\, dt$$

$$+ \frac{1}{2\pi i} \int_{|t-x|>\varepsilon} \frac{t-x}{(t-x)^2+\varepsilon^2} h(t)\, dt.$$

$$= \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon + \mathrm{III}_\varepsilon.$$

Now assume for simplicity that h is a Schwartz space function, $h \in \mathcal{S}(\mathbb{R})$. Then clearly by dominated convergence

(38.2) $$\lim_{\varepsilon \downarrow 0} \mathrm{I}_\varepsilon = \frac{h(x)}{2} \int \frac{1}{\pi(u^2+1)}\, du = \frac{h(x)}{2}$$

On the other hand, by oddness,

$$|\mathrm{II}_\varepsilon| = \left| \frac{1}{2\pi i} \int_{|t-x|<\varepsilon} \frac{t-x}{(t-x)^2+\varepsilon^2} \big(h(t) - h(x) \big)\, dx \right|$$

$$\leq \|h'\|_{L^\infty} \left| \frac{1}{2\pi} \int_{|t-x|<\varepsilon} \frac{(t-x)^2}{(t-x)^2+\varepsilon^2}\, dx \right.$$

$$\leq \|h'\|_\infty \frac{2\varepsilon}{2\pi}$$

$$\to 0 \quad \text{as} \quad \varepsilon \downarrow 0.$$

Finally

$$\mathrm{III}_\varepsilon = \frac{1}{2\pi i} \int_{|t-x|>\varepsilon} \left(\frac{t-x}{(t-x)^2+\varepsilon^2} - \frac{1}{t-x} \right) h(t)\, dt$$

$$+ \frac{1}{2\pi i} \int_{|t-x|>\varepsilon} \frac{h(t)}{t-x}\, dt$$

$$= \mathrm{IV}_\varepsilon + \mathrm{V}_\varepsilon.$$

Now

$$|\mathrm{IV}_\varepsilon| = \frac{1}{2\pi} \left| \int_{|t-x|>\varepsilon} \frac{\varepsilon^2}{(t-x)^2+\varepsilon^2} \frac{h(t)}{t-x}\, dt \right|$$

$$= \frac{1}{2\pi} \left| \int_{|u|>1} \frac{1}{u^2+1} \frac{h(x+\varepsilon u)}{\varepsilon u}\, \varepsilon\, du \right|$$

$$\xrightarrow[\varepsilon \downarrow 0]{} \frac{1}{2\pi} |h(x)| \left| \int_{|u|>1} \frac{du}{u(u^2+1)} \right| = 0$$

Thus we see that for $\Sigma = \mathbb{R}$ and $h \in \mathcal{S}(\mathbb{R})$, say,

$$(40.1) \quad C^+h(x) = \lim_{\varepsilon \downarrow 0} Ch(x+i\varepsilon) = \frac{1}{2}h(x) + \frac{i}{2}Hh(x)$$

where

$$(40.2) \quad Hh(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{|t-x|>\varepsilon} \frac{h(t)}{x-t}\,dt$$

$\equiv$ Hilbert transform of h

Note that $\frac{1}{\pi}\int_{|t-x|>\varepsilon} h(t)/x-t \,dt = \frac{1}{\pi}\int_{|t-x|>1} \frac{h(t)}{x-t}\,dt$
$+\frac{1}{\pi}\int_{\varepsilon<|t-x|<1} (h(t)-h(x))/x-t \,dt \underset{\varepsilon\downarrow 0}{\longrightarrow} \frac{1}{\pi}\int_{|t-x|>1} \frac{h(t)\,dt}{x-t} + \frac{1}{\pi}\int_{0<|t-x|<1} \frac{h(t)-h(x)}{x-t}$
so that $Hh(x) = \lim_{\varepsilon\downarrow 0} I_\varepsilon$ indeed exists ptwise if $h \in \mathcal{S}$.
Similarly one finds

$$(40.3) \quad C^-h(x) = \lim_{\varepsilon \downarrow 0} Ch(x-i\varepsilon) = -\frac{1}{2}h(x) + \frac{i}{2}Hh(x)$$

We see, as noted before, that

$$(40.4) \quad C^+ - C^- = \mathrm{id}$$

and also

$$(40.5) \quad C^+ + C^- = iH$$

The full facts for $\Sigma = \mathbb{R}$ are the following:

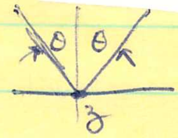
(see eg Katznelson (Harmonic Analysis), Duren (H^p spaces), ..., and see below for some proofs)

(41.1) For $1 \le p < \infty$ and $h \in L^p(\mathbb{R})$,

$$C^{\pm} h(z) = \lim_{z' \to z_\pm} Ch(z') \quad , \quad z \in \mathbb{R}$$

exists as a <u>non-tangential limit</u> for a.e. $z \in \mathbb{R}$,

i for a.e. $z \in \mathbb{R}$ the following is true. For any θ, $0 < \theta < \frac{\pi}{2}$, let $\Gamma_z(\theta)$ be the cone supported at z of opening angle 2θ



Then (for almost all z) the above limits exist as $z' \to z$, $z' \in \Gamma_z(\theta)$.

(41.2) For $1 \le p < \infty$ and $h \in L^p(\mathbb{R})$,

$$Hh(z) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{|z-t| > \varepsilon} \frac{h(t)\,dt}{z-t}$$

exists for a.e. $z \in \mathbb{R}$ and

$$C^{\pm} h(z) = \pm \tfrac{1}{2} h(z) + \tfrac{i}{2} Hh(z) \qquad \text{for a.e. } z \in \mathbb{R}.$$

(41.3) For $1 < p < \infty$,

$$\| Hh \|_{L^p} \le c_p \| h \|_{L^p} \quad , \quad h \in L^p(\mathbb{R})$$

(42.1) In particular this $\Rightarrow$ that $C^{\pm} = \pm\frac{1}{2} + \frac{i}{2} H \in \mathcal{L}(L^p(\mathbb{R}))$, $1<p<\infty$.
Moreover the limit

(42.2) $$H h = \underset{\Sigma}{H} h = \lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{|z-t|>\varepsilon} \frac{h(t)}{z-t}\, dt$$

exists in L^p. The same is true for $C^{\pm} h(z) = \lim_{\varepsilon\downarrow 0} \int \frac{h(s)}{s-(z\pm i\varepsilon)} \frac{ds}{2\pi i}$

The restriction $p>1$ in (41.3) is clear from the following observation: suppose $h \in L^1(\mathbb{R})$ and h has compact support, say $h(x)=0$ for $|x|>L$. Then for $|z|>L$, $Hh(z) = \frac{1}{\pi}\int_{-L}^{L} \frac{h(t)}{z-t}\, dt \sim \frac{1}{z}$ as $|z|\to\infty$, and so $Hh(z) \notin L^1(\mathbb{R}, dz)$. So $1<p<\infty$ is the most we can hope for (<u>Exercise</u>: H does not map $L^\infty \to L^\infty$). <u>Note</u>: However H maps $L^1 \to$ weak L^1 (see refs).

<u>Question</u>: On which (simple, rectifiable) contours $\Sigma \subset \mathbb{C}$ does (41.3) remain true?

Quite remarkably, it turns out that there are <u>necessary</u> and <u>sufficient</u> conditions on a simple rectifiable contour for (41.3) to hold (the result is due to many authors with Guy David making the final decisive contribution). Let Σ be a simple, rectifiable curve in $\mathbb{C}$:

For any $z \in \Sigma$, and any $r > 0$, let

(43.1) $$l_r(z) = \text{arc length of } (\Sigma \cap D_r(z))$$

where $D_r(z)$ = ball of radius r centered at z

Set

(43.2) $$\lambda = \lambda_\Sigma = \sup_{z \in \Sigma,\ r>0} \frac{l_r(z)}{r}$$

Theorem 43.3.

Suppose $\lambda_\Sigma < \infty$. Then for any $1 < p < \infty$, the limit in (42.2) exists and defines a bounded operator

(43.4) $$\| H h \|_{L^p} \le c_p \, \| h \|_{L^p} \quad , \quad h \in L^p, \quad c_p < \infty .$$

Conversely, if the limit in (42.2) exists and defines a bounded operator H in $L^p(\Sigma)$ for some $1<p<\infty$, then the limit exists and gives rise to a bounded operator for all p, $1<p<\infty$, and $\lambda_\Sigma < \infty$.

Moreover if $\lambda_\Sigma < \infty$, then the non-tangential limits $C^\pm h(z)$ in (41.1) $\exists$ for all $1<p<\infty$, as well as the pointwise a.e. limit $Hh(z)$ in (41.2), and necessarily $C^\pm h(z) = \pm\frac{1}{2}h(z) + \frac{i}{2} Hh(z)$, a.e. $z \in \Sigma$. Of course the pointwise limit for $Hh(z)$ agrees a.e. with the L^p limit for $Hh(z)$ in (42.2) for $1<p<\infty$.

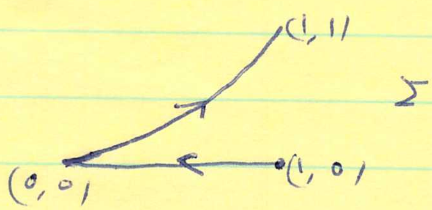
A good reference for the above theorem is

A. Böttcher and Y. I. Karlovich,
Carleson curves, Muckenhoupt weights and Toeplitz operators
Birkhauser, 1997.

A curve for which $\lambda_\Sigma < \infty$ is called <u>A-regular</u> or <u>AD-regular</u> (A = Ahlfors, AD = Ahlfors - David) or a <u>Carleson curve</u>. Theorem 43.3 easily extends to a finite union of AD-regular curves.

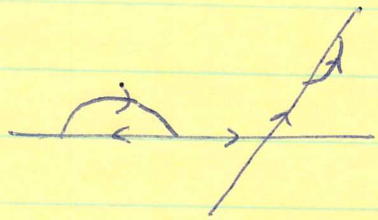
To get some sense of the subtlety of the result, consider the following curve Σ:

$$\Sigma = \{0 \le x \le 1, y = 0\} \cup \{(x, x^2) : 0 \le x \le 1\}$$



Clearly $\lambda_\Sigma < \infty$. <u>Exercise</u> Show directly that H is bounded in $L^2(\Sigma)$.

We will not need the full strength of Theorem (43.3) in this course. In fact it is enough for us to consider curves Σ which are a finite union of straight line segments and subarcs of the circle



For $\Sigma = \mathbb{R}$ or $\Sigma = \{|z|=1\}$ Theorem 43.3 is classical as indicated above (Katznelson, ...). The steps in the proof, more or less, proceed as follows: let $\Sigma = \mathbb{R}$ ($\Sigma = \{|z|=1\}$ is similar).

1) The Fourier transform

$$\mathcal{F} f(z) = \int_{\mathbb{R}} e^{-izt} f(t) \frac{dt}{\sqrt{2\pi}} = \hat{f}$$

$$\mathcal{F}^{-1} f(x) = \int e^{izx} f(z) \frac{dz}{\sqrt{2\pi}} = \check{f}$$

diagonalizes H (exercise),

$$H h = -i \left(\hat{h}\, \mathrm{sgn}(\cdot)\right)^{\vee}, \quad h \in L^2(\mathbb{R})$$

and hence

$$\|H h\|_{L^2} = \|h\|_{L^2}$$

2) $$C^{\pm} h = \frac{1}{2}\left(\pm h + (\hat{h}\, \mathrm{sgn}(\cdot))^{\vee}\right) = \left[\left(\frac{\pm 1 + \mathrm{sgn}(\cdot)}{2}\right)\hat{h}\right]^{\vee}$$

$$= (\chi^+ \hat{h})^{\vee}, \text{ resp } (-\chi^- \hat{h})^{\vee}$$

where χ^+, χ^- are the characteristic functions of $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively. Thus $\pm C^{\pm}$ are the orthogonal projections,

$$\|C^{\pm}\|_{L^2} = 1,$$

onto the <u>Hardy spaces</u> H^2 of functions which are analytic in $\{\Im z > 0\} \equiv \mathbb{C}^+$, $\{\Im z < 0\} \equiv \mathbb{C}^-$ resp, and, one sees (exercise),

$$\sup_{\varepsilon > 0} \int |Ch(x+i\varepsilon)|^2 \, dx < \infty$$

and

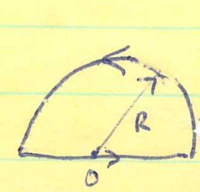
$$\sup_{\varepsilon > 0} \int |Ch(x-i\varepsilon)|^2 \, dx < \infty \quad \text{, resp.}$$

(3) (Following Riesz):
Now suppose $f \in C_0^\infty(\mathbb{R})$ and consider

$$Cf(z) = \int_{\mathbb{R}} \frac{f(t)}{t-z} \frac{dt}{2\pi i}, \quad z \in \mathbb{C}^+$$

which is analytic, $z \in \mathbb{C}^+$.
Then $Cf(z) \sim \frac{1}{z}$ as $z \to \infty$ and the calculations on p 38 et seq show that $Cf(z)$ is continuous down to the real axis. Hence by Cauchy's theorem, for $R > 0$,

$$\int_{C_R} (Cf(z))^4 \, dz = 0$$

where  C_R

and as $R \to \infty$, $\int_{\frown} (Cf(z))^4 \, dz \to 0$.

Hence

$$\int_{-\infty}^{\infty} [(C^+f)(x)]^4 dx = \lim_{R\to\infty} \int_{-R}^{R} (C^+f(x))^4 dx = 0$$

But $C^+f = \frac{1}{2}f + i\frac{Hf}{2}$ and so

(48.1) $$\int_{\mathbb{R}} [f^4 + 4f^3(Hf)i + 6f^2(Hf)^2(i)^2 + 4f(Hf)^3(i^3) + (Hf)^4 i^4] dx = 0.$$

Now suppose f is real valued. Then taking the real part of (48.1) we find

$$\int [f^4 - 6f^2(Hf)^2 + (Hf)^4] dx = 0.$$

$\Rightarrow$

$$\int (Hf)^4 dx \le 6\int f^2(Hf)^2 - \int f^4$$

$$\le 6\left[\int \frac{c}{2} f^4 + \int \frac{1}{2c}(Hf)^4\right] - \int f^4,$$

for any $c > 0$. Take $c = 6$. Then

$$\frac{1}{2}\int (Hf)^4 \le (18-1)\int f^4$$

i.e.

$$\int (Hf)^4 \le 34 \int f^4$$

The case where f is complex valued is handled by taking real and imaginary parts. Thus, by density, H maps $L^4 \to L^4$

boundedly. The same argument works for any pos. even integer p (Exercise). But then the result for all $p \geq 2$ follows by interpolation (Exercise). Now for $p \geq 2$, $L^p(\mathbb{R})$ is dual to $L^q(\mathbb{R})$, $\frac{1}{q} + \frac{1}{p} = 1$, Also a straightforward calculation (exercise) shows that the dual H' of H,

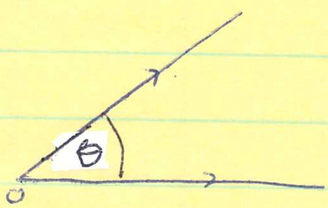
$$Hf(z) = \lim_{\varepsilon \downarrow 0} \int_{|z-s|} \frac{f(s)}{z-s} \frac{ds}{\pi}$$

is just $-H$. But by general theory, H is bounded $\Leftrightarrow$ H' is bounded in the dual space. For $p \geq 2$, we have $1 < q \leq 2$ and so H' is bded in L^q, $1 < q \leq 2$. But $H' = -H$, and we conclude that for $\Sigma = \mathbb{R}$,

$$H \text{ is bded in } L^p, \quad 1 < p < \infty.$$

We need to consider self-intersecting contours. How can we see directly, for example, that if

$$\Sigma_\theta = [(0,\infty)] \cup [e^{i\theta}(0,\infty)] \quad , \quad 0<\theta<2\pi .$$



then $H \in \mathcal{L}(L^2(\Sigma) \to L^2(\Sigma))$? In particular we need to know that if $f \in L^2(0,\infty)$, then

$$(C_\theta f)(r) = \int_0^\infty \frac{f(s)}{s - zr} \frac{ds}{2\pi i} \quad , \quad z = e^{i\theta}$$

lies in $L^2((0,\infty), dr)$ and

(50.1) $$\|C_\theta f\|_{L^2} \leq c_\theta \|f\|_{L^2}$$

As in Beals, D, Tomei, Direct & Inverse Scat. on the line, p88, we use the Mellin transform $\mathcal{M}$:

$$\mathcal{M} : L^2(0,\infty) \to L^2(-\infty,\infty)$$
$$\mathcal{M}f(s) = \int_0^\infty x^{-\frac{1}{2}+is} f(x) \frac{dx}{\sqrt{2\pi}} \quad , \quad f \in L^2(0,\infty),$$
$$\|\mathcal{M}f\|_{L^2(-\infty,\infty)} = \|f\|_{L^2(0,\infty)}$$

(Exercise) $\mathcal{M}$ is a unitary map and

$$\mathcal{M}^{-1}h(x) = \int_{-\infty}^{\infty} h(s)\, x^{-\frac{1}{2}-is}\,\frac{ds}{\sqrt{2\pi}}.$$

The Mellin transform is the Fourier transform associated with the multiplicative group $\mathbb{R}_+$. Moreover this group commutes with $C_\theta f$, i.e. if $T_\lambda f(x) = f(\lambda x)$, $\lambda > 0$ then

$$T_\lambda(C_\theta f) = C_\theta(T_\lambda f)$$

and so C_θ is diagonalized by $\mathcal{M}$. Indeed we see that (exercise)

$$(\mathcal{M} C_\theta \mathcal{M}^{-1} h)(s) = \frac{z^{-\frac{1}{2}+is}}{1+e^{-2\pi s}}\cdot h(s), \qquad h \in L^2(-\infty,\infty)$$

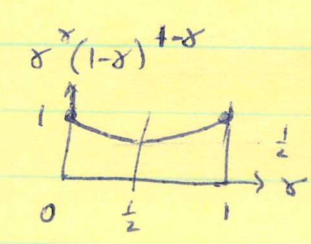
$$= \left(\frac{e^{-i\theta/2}\, e^{-\theta s}}{1+e^{-2\pi s}}\right) h(s).$$

Hence

(51.1) $$\|C_\theta\|_{L^2(0,\infty)} = \sup_{s\in\mathbb{R}} \frac{e^{-\theta s}}{1+e^{-2\pi s}} = \gamma^\gamma(1-\gamma)^{1-\gamma}$$

where

$$\gamma = \frac{\theta}{2\pi}.$$



In particular we see that C_θ is bdd

uniformly for all $0<\theta<2\pi$; the bound is a minim $= \frac{1}{2}$

at $\theta = \pi$ 

and a maxim at $\theta = 0+$ or $2\pi-$

What about $L^p(0,\infty) \to L^p(e^{i\theta}(0,\infty))$, $1<p<\infty$, $0<\theta<2\pi$?
Fix $1<p<\infty$.
For $f, g \in C_0^\infty(0,\infty)$ set

$$h(z) \equiv e^{\frac{i\pi}{p}z} \int_0^\infty dr\, g(r) \int_0^\infty \frac{f(s)}{s - e^{i\pi z} r} \frac{ds}{2\pi i}$$

where $0 < \mathrm{Re}\, z < 1$. Then clearly $h(z)$ is analytic in

the strip $0<\mathrm{Re}\, z<1$. Moreover, $h(z)$ is continuous in

$0 \le \mathrm{Re}\, z \le 1$ and one finds (exercise) for $y \in \mathbb{R}$

$$h(iy) = \lim_{x \downarrow 0} h(x+iy) = e^{-\frac{\pi}{p}y} \int_0^\infty g(r)\, dr \left(C^+ f\right)(r e^{-\pi y})$$

$$h(1+iy) = \lim_{x \uparrow 1} h(x+iy) = e^{-\frac{\pi}{p}y} e^{\frac{i\pi}{p}} \int_0^\infty g(r)\, dr \left(C^+ f\right)(r(-e^{-\pi y}))$$

Thus

$$|h(iy)| \le e^{-\pi y/p} \|g\|_{L^q} \|C^+ f(\cdot\, e^{-\pi y})\|_{L^p}$$

$$= \|g\|_{L^q} \|C^+ f\|_{L^p}$$

$$\le \hat{c}_p \|g\|_{L^q} \|f\|_{L^p} \quad , \text{ where } \hat{c}_p \le \frac{1+c_p}{2}, \ c_p \text{ as in } (43.4)$$

Also

$$|h(1+iy)| \le e^{-\frac{\pi}{p} y} \|g\|_{L^q} \|C^+ f(\cdot(-e^{-\pi y}))\|_{L^q}$$

$$\le \hat{c}_p \|g\|_{L^q} \|f\|_{L^p}.$$

Finally as $\sup_{\operatorname{Im} \zeta > 0} |Cf(\zeta)| < \infty$ (why?), we see that for $y \ge 0$, $0 \le x \le 1$,

$$|h(x+iy)| \le e^{-\pi y/p} \int_0^\infty dr |g(r)| \sup_{\operatorname{Im}\zeta>0} |Cf(\zeta)|$$

$$\le \int_0^\infty dr |g(r)| \sup_{\operatorname{Im}\zeta>0} |Cf(\zeta)|$$

$$< \infty \quad , \text{ as } g \in C_0^\infty(0,\infty).$$

On the other hand, for $y < 0$,

$$h(x+iy) = e^{\frac{i\pi x}{p}} e^{-\pi y/p} \int_0^\infty dr\, g(r) \int_0^\infty \frac{f(s)}{s - r e^{-\pi y} e^{i\pi x}} \frac{ds}{2\pi i}$$

As $g, f \in C_0^\infty(0,\infty)$ it follows (exercise) that for $y < 0$

$$|h(x+iy)| \le c\,\frac{e^{-\pi y/p}}{e^{-\pi y}} = c\,e^{\pi y/q} \le c$$

for some constant $c = c(f, g)$.

We conclude that $h(z)$ is bded in $0 \le \operatorname{Re} z \le 1$ and hence by the Hadamard 3-line th^m, we have

$$|h(z)| \le \left(\tilde{c}_p \|g\|_{L^q} \|f\|_{L^p}\right)^{1-\operatorname{Re} z} \left(\tilde{c}_p \|g\|_{L^q} \|f\|_{L^p}\right)^{\operatorname{Re} z}$$

$$= \tilde{c}_p \|g\|_{L^q} \|f\|_{L^p}, \quad 0 \le \operatorname{Re} z \le 1.$$

In particular for $z = \phi/\pi$, $0 < \phi \le \pi$, we find

$$|h(\phi/\pi)| = \left|\int_0^\infty g(r)\, Ch(re^{i\phi})\,dr\right|$$

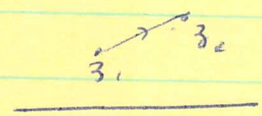
$$\le \tilde{c}_p \|g\|_{L^q} \|h\|_{L^p}.$$

Thus for $0 < \theta \le \pi$, $1 < p < \infty$,

(54.1) $$\|C_\theta h\|_{L^p(dr,(0,\infty))} \le \tilde{c}_p \|h\|_{L^p(0,\infty)}$$

where $\tilde{c}_p$ is indep. of θ. (The case $\pi < \theta < 2\pi$ clearly follows from (54.1) by complex conjugation).

The estimate (54.1) is extremely useful. For example, if $f \in H^1(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R}) \}$, then for $z_1, z_2 \in \mathbb{C}^+$



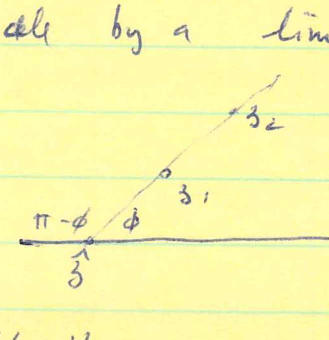
we obtain by integrating along the straight line $z_1 \to z_2$

$$|Cf(z_1) - Cf(z_2)| = \left| \int_{z_1}^{z_2} \frac{d}{dz} Cf(z)\, dz \right|$$

$$= \left| \int_{z_1}^{z_2} (Cf')(z)\, dz \right|$$

$$\leq |z_2 - z_1|^{\frac{1}{2}} \left(\int_{z_1}^{z_2} |Cf'(z)|^2 |dz| \right)^{\frac{1}{2}}$$

Here we used the fact that C commutes with $\frac{d}{dz}$. Now extend $\overrightarrow{z_1 z_2}$ to the real axis (the case where $\overrightarrow{z_1 z_2}$ is parallel to the real axis is easy to handle by a limiting procedure)



We have $\int_{z_1}^{z_2} |Cf'(z)|^2 |dz| = \int_{z_1}^{z_2} |Cf' \chi_{(\hat{z},\infty)} + Cf' \chi_{(-\infty,\hat{z})}|^2 |dz|$

$\leq 2 \int_{z_1}^{z_2} |Cf' \chi_{(\hat{z},\infty)}|^2 |dz| + 2 \int_{z_1}^{z_2} |Cf' \chi_{(-\infty,\hat{z})}|^2 |dz|$

$$\leq 2\left(\|f'\chi_{(\mathfrak{z}',\infty)}\|^2 + \|f'\chi_{(-\infty,\mathfrak{z}')}\|^2\right).$$

$$= 2\,\|f'\|_{L^2}^2$$

Thus

(56.1) $$|Cf(\mathfrak{z}_1) - Cf(\mathfrak{z}_2)| \leq \sqrt{2}\,\|f'\|_{L^2}\,|\mathfrak{z}_1 - \mathfrak{z}_2|^{\frac{1}{2}}.$$

We have proved the following result.

Proposition (56.2)

Suppose $f \in H^1(\mathbb{R})$. Then $Cf(\mathfrak{z})$ is an analytic function in $\mathbb{C}^+$ which extends to $\overline{\mathbb{C}^+}$ continuously as a Hölder-$\frac{1}{2}$ function satisfying (56.1).

Remark There is clearly an $H^1_p(\mathbb{R}) = \{f \in L^p,\ f' \in L^p\}$ version of this result, $1 < p < \infty$ (Exercise!).