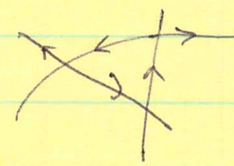
Lecture 4 Reference: D + Zhou, Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space, CPAM, 56 (2003), 1029 - 1077

We now make the notion of a RHP precise.

Let Σ be an oriented contour in $\mathbb{C}$ and let $v : \Sigma \to Gl(n, \mathbb{C})$ be a jump matrix on Σ, $v, v^{-1} \in L^\infty(\Sigma)$.

All the calculations that follow go through for contours that are a finite union of A-regular simple curves with only a finite # of pts of intersection. But if you want you can just think of Σ of a union of line segments and sub-arcs of the unit circle.



Σ. Introduce the Cauchy operators on Σ,

$$Ch(z) = C_\Sigma h(z) = \int_\Sigma \frac{h(s)}{s-z} \frac{ds}{2\pi i}, \quad z \in \mathbb{C} \setminus \Sigma$$

and $h \in L^p(\Sigma)$, $1 < p < \infty$. Also $C^\pm h(z) = \lim_{z' \to z_\pm} Ch(z)$, $z \in \Sigma$.

$= \pm \frac{h}{2} + \frac{i}{2} Hh$, $H =$ Hilbert transform.

We say that a pair of $L^p(\Sigma)$ functions $f_\pm \in \partial C(L^p)$

if $\exists$ a (unique) function $h \in L^p(\Sigma)$ such that

(58.1) $$f_\pm(z) = (C^\pm h)(z), \quad z \in \Sigma$$

In turn we call $f(z) = Ch(z)$, $z \in \mathbb{C} \setminus \Sigma$, the

<u>extension</u> of $f_\pm = C^\pm h \in \partial C(L^p)$ off Σ.

<u>Definition 58.2</u> Fix $1 < p < \infty$. Given Σ, v and a mble

function f, we say that $m_\pm \in f + \partial C(L^p)$ solves

an <u>inhomogeneous RHP of the first kind</u> $(IRHP1_{L^p})$ if

(58.3) $$m_+(z) = m_-(z)\, v(z), \quad z \in \Sigma.$$

<u>Definition 58.4</u> Fix $1 < p < \infty$. Given Σ, v and a

function $F \in L^p(\Sigma)$, we say that $M_\pm \in \partial C(L^p)$

solves an <u>inhomogeneous RHP of the second kind</u>

$(IRHP2_{L^p})$ if

(58.5) $$M_+(z) = M_-(z)\, v(z) + F(z), \quad z \in \Sigma.$$

Recall that m solves the normalized RHP (Σ, v) if, at least formally,

- $m(z)$ is analytic in $\mathbb{C} \setminus \Sigma$
- $m_+(z) = m_-(z)\, v(z)$, $z \in \Sigma$
- $m(z) \to I$ as $z \to \infty$.

More precisely, we make the following definition:

<u>Definition 59.1</u> Fix $1 < p < \infty$. We say that $m_\pm$ solves the normalized RHP $(\Sigma, v)_{L^p}$ if $m_\pm$ solves the IRHP1$_{L^p}$ with $f \equiv I$.

In the above definition, if $m_\pm - I = C^\pm h$, then clearly the extension $m(z) = I + C h(z)$ of $m_\pm$ off Σ solves the RHP in the formal sense. If $p = 2$, which is the case of most interest, we will drop the subscript and simply write IRHP1, IRHP2 and (Σ, v). Let

(59.2) $$v = (v^-)^{-1} v^+ = (I - w^-)^{-1} (I + w^+)$$

be a pointwise a.e. factorization of v with

$v^\pm$, $(v^\pm)^{-1} \in L^\infty$, and let C_w, $w = (w^-, w^+)$

denote the basic associated operator

(60.1) $$C_w h = C^+(hw^-) + C^-(hw^+)$$

acting on $L^p(\Sigma)$ - matrix-valued functions. As $w^\pm \in L^\infty$, $C_w \in \mathcal{L}(L^p)$ for all $1 < p < \infty$.

The utility of $IRHP1_{L^p}$ and $IRHP2_{L^p}$ will soon become clear:

$$IRHP1_{L^p} \equiv IRHP2_{L^p}$$

$\updownarrow$ inverse of $(1 - C_w)$

$\updownarrow$ useful for deformations of RHP in $\mathbb{C}$

Now suppose f and v are such that $f(v - I) \in L^p(\Sigma)$ for some $1 < p < \infty$. Let $\mu_\pm$ solve the $IRHP2_{L^p}$ with $F = f(v - I)$. Then for

$$m_\pm = \mu_\pm + f$$

we have

$$m_+ = M_+ + f$$

$$= M_- v + F + f \quad = M_- v + f(v - I) + f.$$

$$= M_- v + f v \quad = (M_- + f) v = m_- v$$

so $m_\pm$ solves solves the IRHP1$_{L^p}$ with $m_\pm - f \in \partial C(L^p)$.

If we reverse the argument, we see that if $m_\pm$ solves the IRHP1$_{L^p}$ with any f, then $M_\pm = m_\pm - f$ solves the IRHP2$_{L^p}$ with F of the form $f(v-I)$ which, however, is not the general case since $v - I$ need not be invertible. To prove the equivalence of IRHP1$_{L^p}$ and IRHP2$_{L^p}$, we must proceed in a different way.

Given $F \in L^p(\Sigma)$, let $m_\pm$ solve IRHP1$_{L^p}$ with $f = C^- F$. By assumption, $m_\pm - f = C^\pm h$ for some $h \in L^p(\Sigma)$, and hence.

$$m_- = f + C^- h = C^-(F+h)$$

$$m_+ + F = f + C^+ h + (C^+ - C^-)F$$

$$= C^- F + C^+(h+F) - C^- F$$

$$= C^+(h+F)$$

It follows that

$$M_+ = m_+ + F, \quad M_- = m_-$$

solve IRHP2 with the given F,

$$M_+ = m_+ + F = m_- v + F$$
$$= M_- v + F$$

We have proved the following result:

<u>Thm 62.1</u> ($IRHP1_{L^p} \equiv IRHP2_{L^p}$)

Suppose f and v are such that $f(v-I) \in L^p(\Sigma)$

for some $1<p<\infty$. Then

(62.2) $$m_\pm = M_\pm + f$$

solves $IRHP1_{L^p}$ if $M_\pm$ solves $IRHP2_{L^p}$ with

$F = f(v-I)$. Conversely, if $F \in L^p(\Sigma)$, then

(63.1) $$H_+ = m_+ + F, \quad H_- = m_-$$

solves $IRHP2_{L^p}$ if $m_\pm$ solves $IRHP1_{L^p}$

with $f = C^- F$.

We are principally interested in the case where $f \in L^\infty$

and $v - I \in L^p(\Sigma)$ for some $1 < p < \infty$.

To establish the connection with the inverse of

$1 - C_\omega$, let $m_\pm$ solve $IRHP1_{L^p}$ with $f \in L^p(\Sigma)$.

Thus $m_\pm = f + C^\pm h$ for some $h \in L^p(\Sigma)$. Also

$m_+ = m_- v = m_-(v^-)^{-1} v^+$. Set $\mu = m_-(v^-)^{-1} = m_+(v^+)^{-1} \in L^p$.

and define $H(z) = \big(C(\mu(\omega^+ + \omega^-))\big)(z)$, $z \in \mathbb{C} \setminus \Sigma$.

Then we have on Σ

$$H_+ = C^+ \mu(\omega^+ + \omega^-) = C^+ \mu\omega^- + C^+ \mu\omega^+$$

$$= C^+ \mu\omega^- + C^- \mu\omega^+ + \mu\omega^+ , \text{ as } C^+ - C^- = id,$$

$$= C_\omega \mu + \mu\omega^+ = (C_\omega - 1)\mu + \mu(I + \omega^+)$$

$= (C_\omega - 1)\mu + \mu v^+ = ((C_\omega - 1)\mu + m_+$. Similarly

$H_- = ((C_\omega - 1)\mu + m_-$ or

$$m_\pm - f - H_\pm = (1 - C_\omega)\mu - f$$

But

$$m_\pm - f - H_\pm \in \partial C(L^p)$$

and hence.

$$(1 - C_\omega)\mu = f$$

Conversely, if $\mu \in L^p(\Sigma)$ solves $(1 - C_\omega)\mu = f$, then the above calculation shows that $H \equiv C(\mu(\omega^+ + \omega^-))$ satisfies $H_\pm = -f + \mu v^\pm$. Thus setting $m_\pm = \mu v^\pm$, we see that $m_+ = m_- v$ and $m_\pm - f \in \partial C(L^p)$.

In particular $\mu \in L^p$ solves $(1 - C_\omega)\mu = 0 \Leftrightarrow m_\pm = \mu v^\pm$ solves the homogeneous RHP

$$m_+ = m_- v, \quad m_\pm \in \partial C(L^p). \tag{64.1}$$

We summarize the above calculations as follows.

Proposition 65.1 Let $1 < p < \infty$. Then

$1 - C_\omega$ is a bijection in $L^p(\Sigma)$

$\Leftrightarrow$ IRHP1$_{L^p}$ has a unique solution for all $f \in L^p(\Sigma)$

$\Leftrightarrow$ IRHP2$_{L^p}$ has a unique solution for all $F \in L^p(\Sigma)$

Moreover, if $(1-C_\omega)^{-1}$ $\exists$, then for all $f \in L^p(\Sigma)$

(65.1) $$(1 - C_\omega)^{-1} f = m_+(v^+)^{-1} = m_-(v^-)^{-1}$$

$$= (M_+ + f)(v^+)^{-1} = (M_- + f)(v^-)^{-1}$$

where $m_\pm$ solves IRHP1$_{L^p}$ with the given f and $M_\pm$ solves IRHP2$_{L^p}$ with $F = f(v - I)$. Conversely if $M_\pm$ solves IRHP2$_{L^p}$ with $F \in L^p(\Sigma)$, then

(65.2) $M_+ = ((1 - C_\omega)^{-1}(C^- F))v^+ + F$ and $M_- = ((1 - C_\omega)^{-1} C^- F)v^-$.

Finally, if $f \in L^\infty(\Sigma)$ and $v^\pm - I \in L^p(\Sigma)$, then (65.1) remains valid provided we interpret

(65.3) $$(1 - C_\omega)^{-1} f \equiv f + (1 - C_\omega)^{-1}(C_\omega f).$$

This is true in particular for the normalized RHP $(\Sigma, v)_p$ where $f \equiv I$.