Lecture 5

<u>Remark</u>: If we take $v^+ = v$, $v^- = I$ then

(66.1) $$C_w h = C^-(h(v - I))$$

which is the operator C_v introduced on p31.

The above Prop $\Rightarrow$, in particular, that if $\mu \in I + L^p$ solves

(66.2) $$(1 - C_w)\mu = I$$

as in (65.3), then $m_\pm = \mu v^\pm$ solves the normalized RHP (Defn. 59.1).

The fact the $IRHP1_{L^p}$ and $IRHP2_{L^p}$ depend on v and not on the particular factorization $v = (v^-)^{-1} v^+$ (which we are <u>free</u> to choose, for any particular purpose at hand), has the following immediate consequence.

<u>Corollary 66.3</u> Suppose $1 < p < \infty$. The operator $1 - C_w$ is

bijective in $L^p(\Sigma)$ for all factorizations

$v = (v^-)^{-1} v^+ = (I - w^-)^{-1}(I + w^+)$ iff $1 - C_{w'}$ is bijective

for at least one factorization $v = (v'^-)^{-1} v'^+ = (I - w'^-)^{-1}(I + w'^+)$

Moreover, for $f \in L^p(\Sigma)$,

$$(67.1) \quad (1 - C_w)^{-1} f = ((1 - C_{w'})^{-1} f)\, b \quad \text{where} \quad b = v'^+(v^+)^{-1} = v'^-(v^-)^{-1}.$$

Proof:

Suppose $(1-C_w)^{-1}$ $\exists$ in L^p. Then $\mathrm{IRHP1}_{L^p}$ ~~with the given f~~ for any $f \in L^p$, and hence for the given f, has a unique solution $m_\pm$ and by (65.1)

$$(1 - C_w)^{-1} f = m_+ (v^+)^{-1}$$

But as the $\mathrm{IRHP1}_{L^p}$ has a unique solution for any f, $(1 - C_{w'})^{-1}$ $\exists$ in L^p and

$$(1 - C_{w'})^{-1} f = m_+ (v'^+)^{-1}$$

where $m_\pm$ is the (same) solution of the $\mathrm{IRHP1}_{L^p}$ with the given f. Thus

$$(1 - C_w)^{-1} f = m_+ (v'^+)^{-1} \underbrace{v'^+ (v^+)^{-1}} = ((1 - C_{w'})^{-1} f)\, b. \quad \blacksquare$$

Finally we consider uniqueness for the solution of the normalized RHP $(\Sigma, v)_p$ as given in Defn. 59.1. Observe first that if $F(z) = (Cf)(z)$ for $f \in L^p(\Sigma)$ and $G(z) = (Cg)(z)$ for $g \in L^q(\Sigma)$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \le 1$, $1 < p, q < \infty$, then a simple computation shows that

(68.1) $$FG(z) = Ch(z)$$

where

(68.2) $$h(s) = -\frac{1}{2i}\Big(g(s)\,(Hf)(s) + f(s)\,(Hg)(s)\Big)$$

where $Hf(s) =$ Hilbert transform $= \lim_{\varepsilon \downarrow 0} \int_{|s'-s|>\varepsilon} \frac{f(s')}{s-s'} \frac{ds'}{\pi}$ and similarly for $Hg(s)$. Because h clearly lies in $L^r(\Sigma)$, $r \ge 1$, it follows that

(68.3) $$F_+G_+(z) - F_-G_-(z) = h(z)$$

for a.e. $z \in \Sigma$.

<u>Thm 68.3</u> Fix $1 < p < \infty$. Suppose $m_\pm$ solves the normalized RHP $(\Sigma, v)_{L^p}$. Suppose that $m_\pm^{-1}$ exist a.e. on

Σ and $\qquad m_\pm^{-1} \in I + \partial C(L^q)$, $1<q<\infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \leq 1$.

Then the solution of the normalized RHP (Σ, v) is unique.

Proof Suppose $\hat{m}_\pm = I + C^\pm \hat{h}$, $\hat{h} \in L^{\hat{p}}(\Sigma)$ is a second solution of the normalized RHP. We have by assumption $m_\pm^{-1} = I + C^\pm k$ for some $k \in L^q(\Sigma)$. (It is an exercise to show that $I + Ck(z)$, the extension of $m_\pm^{-1}$ to $\mathbb{C} \setminus \Sigma$, is in fact $m(z)^{-1}$). Then arguing as above we see that

$$\hat{m}_\pm m_\pm^{-1} - I = (\hat{m}_\pm - I)(m_\pm^{-1} - I) + (\hat{m}_\pm - I) + (m_\pm^{-1} - I)$$

$$= C^\pm h$$

for some $h \in L^r(\Sigma) + L^p(\Sigma) + L^q(\Sigma)$. Hence

$$\hat{m}_+ m_+^{-1} - \hat{m}_- m_-^{-1} = h$$

But $\hat{m}_+ m_+^{-1} = (\hat{m}_- v)(m_- v)^{-1} = \hat{m}_- m_-^{-1}$, and so $h = 0$

Thus $\hat{m}_\pm m_\pm^{-1} - I = 0$ or $\hat{m}_\pm = m_\pm$.

__Remark__ In the above proof we did not need to assume $\hat{m}_\pm^{-1}$ $\exists$.

__Th^m 70.1__ (Special case $n=2$, $p=2$)

If $n=2$, $p=2$ and $\det v(z) = 1$ a.e. on Σ, then the solution of the normalized RHP $(\Sigma, v) = (\Sigma, v)_{L^2}$ is unique.

__Proof__: Because $n=2$ and $p=2$, (68.1)(68.2) $\Rightarrow$

$(\det m(z))_\pm = 1 + C^\pm h$, where $h \in L^1(\Sigma) + L^2(\Sigma)$ and

$(\det m)_+ - (\det m)_- = h(z)$ a.e. But $(\det m)_+ = (\det m)_-$, as $\det v = 1$, and so $h \equiv 0$. But then $\det m(z)_\pm = 1$.

Hence, if $m_\pm = \begin{pmatrix} m_{11\pm} & m_{12\pm} \\ m_{21\pm} & m_{22\pm} \end{pmatrix}$, we have $m_\pm^{-1} = \begin{pmatrix} m_{22\pm} & -m_{12\pm} \\ -m_{21\pm} & m_{11\pm} \end{pmatrix}$

and so clearly $m_\pm^{-1} \in I + \partial C(L^2)$. The result now follows from Th^m 68.3.

The above results immediately imply that the normalized RHP $(\Sigma = \mathbb{R}, v_{x,t})$ on p17 has a unique solution in $L^2(\mathbb{R})$. Here

$$v_{x,t}(z) = \begin{pmatrix} 1 - |r(z)|^2 & -\bar{r}(z)\, e^{-2i\theta} \\ r(z)\, e^{2i\theta} & 1 \end{pmatrix}, \quad \|r\|_\infty < 1,$$

where $\quad \theta = 4tz^3 + xz, \quad x, t \in \mathbb{R}$

Factorize

$$v_{x,t} = (v_{x,t}^-)^{-1}(v_{x,t}^+) = (I - w_{x,t}^-)^{-1}(I + w_{x,t}^+)$$

$$= \begin{pmatrix} 1 & \bar{r} e^{-2i\theta} \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ r e^{2i\theta} & 1 \end{pmatrix}$$

so that

$$w_{x,t} = (w_{x,t}^-, w_{x,t}^+) = \left(\begin{pmatrix} 0 & -\bar{r} e^{-2i\theta} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ r e^{2i\theta} & 0 \end{pmatrix} \right)$$

Then for $\quad h = (h_{ij})_{1 \le i, j \le 2}$

$$C_{w_{x,t}} h = C^+ h \begin{pmatrix} 0 & -\bar{r} e^{2i\theta} \\ 0 & 0 \end{pmatrix} + C^- h \begin{pmatrix} 0 & 0 \\ r e^{2i\theta} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} C^-(h_{12}\, r e^{i\theta}) & C^+(h_{11}(-\bar{r} e^{2i\theta})) \\ C^-(h_{22}\, r e^{i\theta}) & C^+(h_{21}(-\bar{r} e^{2i\theta})) \end{pmatrix}$$

But for $\Sigma = \mathbb{R}$, C^+ and $-C^-$ are orthogonal projections (exercise) and so $\|C^\pm\|_{L^2(\mathbb{R})} = 1$.

Thus if we use the Hilbert Schmidt norm on matrix, $M = (M_{ij})$,

$$\|M\| \equiv \left(\sum_{i,j} |M_{ij}|^2 \right)^{\frac{1}{2}},$$

we have

$$\|C_{w_{x,t}} h\|_{L^2}^2 = \|C^-(h_{12}\, r e^{i\theta})\|_{L^2(\mathbb{R})}^2 + \|C^-(h_{22}\, r e^{i\theta})\|_{L^2(\mathbb{R})}^2$$

$$+ \|C^+(h_{11}(-\bar{r} e^{-2i\theta}))\|_{L^2(\mathbb{R})}^2 + \|C^+ h_{21}(-\bar{r} e^{-2i\theta})\|_{L^2(\mathbb{R})}^2$$

$$\leq \|r\|_\infty^2 \left(\|h_{11}\|_{L^2}^2 + \|h_{12}\|_{L^2}^2 + \|h_{21}\|^2 + \|h\|_{22}^2 \right)$$

$$= \|r\|_\infty^2 \|h\|_{L^2}^2$$

and so

(72.1) $$\|C_{w_{x,t}}\| \leq \|r\|_\infty < 1$$

It follows that for each $x, t \in \mathbb{R}$, $(1 - C_{w_{x,t}})^{-1}$ $\exists$ in $L^2(\mathbb{R})$ and

(72.2) $$\|(1 - C_{w_{x,t}})^{-1}\|_{L^2} \leq \frac{1}{1 - \|r\|_\infty}$$

Note: The norm of $C_{w_{x,t}}$ is also < 1 if we use the $\ell^2 \times \sup$ norm $\|h\| = \sup_{1 \le i,j \le 2} \|h_{ij}\|_{L^2}$.
This immediately implies the following result.

Th^m 73.1

The normalized RHP $(\Sigma = \mathbb{R}, v_{x,t})$ has a unique solution $m_\pm = I + \partial C(L^2)$ for each $x, t \in \mathbb{R}$.

Moreover,

(73.2) $$m_\pm = I + C^\pm\left(\mu\,(w^+_{x,t} + w^-_{x,t})\right)$$

where $\mu \in I + L^2(\mathbb{R})$ is the unique solution of

(73.3) $$(1 - C_{w_{x,t}})\,\mu = I$$

Remark: As $n = 2$, $p = 2$ and $\det v_{x,t} = 1$, uniqueness in Th^m 73.1 also follows from Th^m 70.1.

We are also interested in solutions of RHP's in senses other than L^p. We say that $m(z)$, $z \in \mathbb{C} \setminus \Sigma$ solves the normalized RHP (Σ, v) in the classical sense if

$$
(74) \begin{cases} \text{(a) } m(z) \text{ is analytic in } \mathbb{C}\setminus\Sigma \text{ and continuous up to} \\ \quad \text{the boundary in each component of } \mathbb{C}\setminus\Sigma \\ \text{(b) } m_+(z) = m_-(z)\, v(z), \quad z \in \Sigma \\ \text{(c) } m(z) \to I \text{ as } z \to \infty \text{ in } \mathbb{C}\setminus\Sigma \end{cases}
$$

All the above limits are taken in the classical sense e.g. (c) means that given $\varepsilon > 0$, $\exists R$ st $|m(z) - I| < \varepsilon$ if $|z| > R$, $z \in \mathbb{C}\setminus\Sigma$.

<u>Note:</u> (Σ, v) has a classical solution, and $\det m_\pm(z) \neq 0$, $\Rightarrow$ $v(z)$ is continuous on Σ.

We consider some standard ways in RHP's arise.

First we consider a remarkable class of operators whose resolvent can be computed in terms of a RHP. These are the so-called <u>integrable operators</u>

(Ref P. Deift "Integrable Ops" AMS Transl. (2) 189 (1999) 69–84) Let Σ be an oriented contour in $\mathbb{C}$. We say that an operator K acting in $L^2(\Sigma)$ is integrable if it has a kernel of the form

(75.1) $$K(z,z') = \frac{\sum_{j=1}^{N} f_j(z)\, g_j(z')}{z - z'}, \quad z, z' \in \Sigma$$

for some functions f_i, g_j, $1 \le i, j \le N$. Integ op's were first singled out as a distinguished class of operators by Sakhnovich in the late 1960's, and ~~their~~ theory was developed fully by Its, Izergin, Kapaev & Slavnov in the early 1990's. Particular examples of such operators had appeared earlier (Tracy, ...) ~ 1960.

The action of K in $L^2(\Sigma)$ is given by

$$Kh(z) = \pi i \sum_{j=1}^{N} f_j(z) \left(\hat{H}(h g_j)\right)(z), \quad z \in \Sigma, \; h \in L^2(\Sigma).$$

where $\hat{H}$ = Cauchy Principal Value operator, $\hat{H} = \frac{1}{i} H$ ← Hilbert transform

$$(\hat{H}\tilde{h})(z) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi i} \int_{\{z' \in \Sigma : |z'-z| > \varepsilon\}} \frac{\tilde{h}(z')}{z - z'} \, dz'$$

for $\tilde{h} \in L^2(\Sigma)$. Thus we see that if Σ is A-regular (which we always assume) and if the functions $f_i, g_j \in L^\infty(\Sigma)$, $1 \le i, j \le N$, then K gives rise to a bdd operator in L^2 (in fact in L^p, $1 < p < \infty$).

Integ. operators indeed have many remarkable properties. To understand the origin of these properties, it is useful to consider a more general situation. Let $\mathcal{F}$ be the space of finite rank op's in a Hilbert space $\mathcal{H}$. Let A be a fixed oper. acting on $\mathcal{H}$, and let $\mathcal{F}_A$ be the class of op's which commutator with A is "small", more precisely, finite rank ie.

$$\mathcal{F}_A = \{ K : [A, K] \in \mathcal{F} \}$$, $\mathcal{F}$ = finite rank operators,

then

77 (1) $\mathcal{F}_A$ is an algebra i.e. if $K_1, K_2 \in \mathcal{F}_A$ & λ_1, λ_2 are scalars, then $\lambda_1 K_1 + \lambda_2 K_2 \in \mathcal{F}_A$, and $K_1, K_2 \in \mathcal{F}_A$.

77 (2) If $K \in \mathcal{F}_A$, and $1-K$ is invertible, then $R \in \mathcal{F}_A$ where $(1-K)^{-1} = 1 + R$.

(1) follows immediately from the identity $[A, K_1 K_2]$ $= K_1 [A, K_2] + [A, K_1] K_2$

To prove (2) use the identity

(77.3) $$[A, (1-K)^{-1}] = (1-K)^{-1} [A, K] (1-K)^{-1}.$$

Observe that if $H = L^2(\Sigma)$ for some oriented $\Sigma \subset \mathbb{C}$, and if A is mult. by z in H,

$$Ah(z) = z h(z) \quad , \quad h \in H$$

Then a simple computation shows that the integ. ops in $\mathcal{H}$ are precisely the kernel operators in $\mathcal{I}_A$.

Suppose K is an integ. op. with kernel

$$\frac{\sum_{i=1}^{n} f_i(z)\, g_i(z')}{z - z'}$$ as in (75.1), and that $(1-K)^{-1}$ ∃:

Suppose further that $R = (1-K)^{-1} - 1$ is a kernel operator. Then we learn from 77(2) and (77.3) that R is also an integrable operator with kernel

(78.1) $$R(z, z') = \frac{\sum_{i=1}^{N} F_i(z)\, G_i(z')}{z - z'}$$

where

(78.2) $$F_i = (1-K)^{-1} f_i, \quad G_i = (1-K^T)^{-1} g_i, \quad 1 \le i \le N.$$

It is a remarkable fact that these functions can be computed in terms of canonical, auxiliary RHP's.

We need an additional algebraic fact. If X_1 and X_2 are Banach spaces, then the <u>commutation formula</u>

$$(79.1)\qquad \frac{\lambda}{DE+\lambda} + D\frac{1}{ED+\lambda}E = 1$$

holds for all bounded operators $D: X_1 \to X_2$, $E: X_2 \to X_1$, in the sense that if $-\lambda \neq 0$ lies in the resolvent set of ED, then $-\lambda$ lies in the resolvent set of DE and $(DE+\lambda)^{-1} = \frac{1}{\lambda}\left(1 - D\frac{1}{ED+\lambda}E\right)$, and vice versa (<u>Exercise</u>: prove (79.1)). The commut. formula has many applications in math and phys see eg. [D], A Commutation formula, Duke Math J. 1978). We now apply this formula to integrable ops.

Let K be an integ. op. on a contour Σ as above. Let $f = (f_1, \ldots, f_N)^T$, $g = (g_1, \ldots, g_N)^T$; let

R_f be the map of right mult. by f, taking a row N-vector functions to scalar functions,

(80.1) $$R_f h(z) = h(z) f(z) = \sum_{i=1}^{N} h_i(z) f_i(z),$$

$$h = (h_1, \dots, h_N),$$

and let R_{g^T} denote the map of right multiplication by the row vector g^T taking scalar functions k to row N-vector functions

(80.2) $$(R_{g^T} k)(z) = k(z)\, g^T(z) = \left(k(z) g_1(z), \dots, k(z) g_N(z)\right)$$

In this notation, K takes the form

(80.3) $$Kk = i\pi R_f \left(\hat{H}(R_{g^T} k)\right) = (DE)(k)$$

where $D = R_f$, $E = i\pi \hat{H} R_{g^T}$. Then ED is a map from row N-vector functions to row N-vector functions

(80.3) $$(ED)(u) = i\pi \hat{H} R_{g^T} R_f u = i\pi \hat{H}(u f g^T)$$

Recalling $C^+ + C^- = iH = -\hat{H}$, we obtain

(81.1) $$EDu = C^+(u(-i\pi f g^T)) + C^-(u(-i\pi f g^T))$$

Thus ED is <u>precisely</u> the singular operator C_ω with $\omega_+ = \omega_- = -i\pi f g^T$. Thus $v_+ = I - i\pi f g^T$, $v_- = I + i\pi f g^T$ and so provided that $1 + i\pi \langle g, f\rangle \neq 0$,

(81.2) $$v = v_-^{-1} v_+ = (I + i\pi f g^T)^{-1} (I - i\pi f g^T) = I - \frac{2i\pi}{1 + i\pi\langle g, f\rangle} f g^T$$

where $\langle g, f\rangle = \sum_{i=1}^N g_i f_i$. Thus the op. K is intimately connected to a RHP (Σ, v) with v given by (81.2). Also by (79.1), $1-K$ is invertible $\Leftrightarrow$ $1 - C_\omega$ is invertible.

We now compute $F_j = (1-K)^{-1} f_j$, $1 \leq j \leq N$ in terms of the solution m of the normalized RHP (Σ, v).

We have by (79.1) for $K = DE$

$$F = (F_1, \ldots, F_N)^T = (1-K)^{-1} f.$$

$$= f + R_f (1 - C_\omega)^{-1} i\pi \hat{H} R_{g^T} f$$

$$= f + R_f (1 - C_\omega)^{-1} i\pi \hat{H} f g^T$$

$$= f + R_f (1 - C_w)^{-1} (C_w I)$$

$$= f + R_f(-I) + R_f (1 - C_w)^{-1} I$$

$$= R_f \mu , \qquad \text{where} \qquad (1 - C_w)\mu = I ,$$

$$= m_\pm v_\pm^{-1} f$$

$$= m_\pm \left(I \mp i\pi f g^T\right)^{-1} f .$$

i.e.

(82.1) $$F = (I \mp i\pi \langle g, f \rangle)^{-1} m_\pm f$$

Similarly (exercise)

(82.2) $$G = (G_1, \ldots, G_N)^T = (I \mp i\pi \langle g, f \rangle)^{-1} \tilde{m}_\pm g$$

where $\tilde{m}$ is the solution of the ~~Z~~ normalized RHP

$(\Sigma, \tilde{v})$ where

(82.3) $$\tilde{v} = I + \frac{2\pi i}{1 - i\pi \langle g, f \rangle} g f^T$$

Observe that $\tilde{v} = (v^T)^{-1}$ and hence $(m_+^T)^{-1} = (m_-^T)^{-1} (v^T)^{-1}$

$= (m_-^T)^{-1} \tilde{v}$. As $(m^T)^{-1} \to I$ as $z \to \infty$ we see that

(83.1) $$\tilde{m}_\pm = (m_\pm^T)^{-1}$$

by uniqueness for the normalized RHP $(\Sigma, \tilde{v})$. This means that to construct F and G, and hence $(1-K)^{-1}$, it is suff. only to consider the RHP (Σ, v).

To summarize: in the special case $\mathcal{H} = L^2(\Sigma)$, with $A \equiv$ mult. by z, we have for K as above:

(83.2) $$R \equiv (1-K)^{-1} - 1 = \frac{\sum_{j=1}^{M} F_j(z) G_j(z')}{z - z'}$$

where

(83.3)
$$F = (F_1, \ldots, F_M)^T = (I \mp i\pi\langle g, f\rangle)^{-1} m_\pm f$$
$$G = (G_1, \ldots, G_M)^T = (I \pm i\pi\langle g, f\rangle)^{-1} (m^T)^{-1}_\pm g$$

where m is the solution of the normalized RHP (Σ, v) with

(83.4) $$v = I - \left(\frac{2\pi i}{1 + i\pi\langle g, f\rangle}\right) f g^T$$

Remark: The above calculations are formal, but can clearly be made rigorous in terms of the precise meaning of a RHP, etc.

Of course, all we have done so far is to replace one sing. integ. op. K with another sing. integ. op C_w. But $C_w \equiv$ RHP's, which are now very well understood!

In almost all cases of interest $\langle f(z), g(z) \rangle = 0$, $z \in \Sigma$, & so

(84.1)
$$\begin{cases} v = 1 - 2\pi i f g^T \\ F = m_\pm f, \quad G = (m_\pm^{-1})^T g \end{cases}$$

Note:
$$m_+ f = m_- v f = m_-(f - 2\pi i f \langle g, f \rangle) = m_- f$$
$$m_+^{-T} f = m_-^{-T} v^{-T} f = m_-^{-T}(I + 2\pi i g f^T) g$$
$$= m_-^{-T} g$$

Note: $v \to \tilde v$ corresponds to $K = \dfrac{\sum f_i(x) g_i(y)}{x-y} \to \tilde K = -\dfrac{\sum g_i(x) f_i(y)}{x-y} = K^T$. Thus $(1-K)^{-1} \exists$ in L^p $\Leftrightarrow (1-\tilde K)^{-1} \exists$ in L^q, $\frac{1}{p}+\frac{1}{q}=1$. Thus, in particular, if $(1-K)^{-1} \exists$, both (Σ, v) and $(\Sigma, \tilde v)$ have normalized solutions. It is surprising that they are so simply related, $\tilde m = (m^T)^{-1}$.