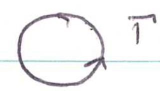
Lecture 6

We now consider some examples of how integrable operators come up naturally (P.D. Integrable operators, AMS Transl. (2) 189 (1999), 69–84)

Exple 1 (Toeplitz determinants)

Let $\varphi(z)$ be a strictly positive, continuous weight on the unit circle $T = \{z : |z| = 1\}$



with Fourier coefficients

$$\varphi_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ij\theta} \varphi(e^{i\theta})\, d\theta, \quad j \in \mathbb{Z}.$$

For any $n \geq 0$,

(85.1) $$T_n = \{\varphi_{j-k}\}_{0 \leq j,k \leq n}$$

is the $(n+1)\times(n+1)$ Toeplitz matrix associated with φ and

(85.2) $$D_n = \det T_n$$

is the Toeplitz determinant associated with φ. It is of great interest to evaluate D_n as $n \to \infty$:

Th^m (Szegö strong limit Theorem)

Let $\varphi(z) = e^{L(z)} \in L^1(\mathbb{T})$ where $\sum_{k=1}^{\infty} k|L_k|^2 < \infty$.

Then as $n \to \infty$

(86.1) $$D_n = e^{(n+1)L_0 + \sum_1^\infty k|L_k|^2}\left(1 + o(1)\right).$$

(See Simon O.P.'s on the unit circle, Chapter 6, for a full discussion with many proofs).

We will eventually show how to prove this Theorem under stronger assumptions on $\varphi(z)$, using RH techniques.

The connection to RHP's is obtained as follows:

Let e_k, $0 \le k \le n$, be the standard basis in $\mathbb{C}^{n+1}$. Then the map

$$U_n : e_k \mapsto z^k \quad , \quad 0 \le k \le n \quad , \quad z \in \mathbb{T}$$

takes $\mathbb{C}^{n+1}$ onto the trigonometric polynomials $\mathcal{P}_n = \{ \sum_{j=0}^n a_j z^j \}$ of degree n and induces a map

$$\tau_n : \mathcal{P}_n \to \mathcal{P}_n$$

which is conjugate to T_n

(87.1) $$\tau_n z^k = U_n T_n U_n^{-1} z^k$$

$$= U_n T_n e_k$$

$$= U_n \sum_{j=0}^{n} \varphi_{j-k} e_j$$

$$= \sum_{j=0}^{n} \varphi_{j-k} z^j , \qquad 0 \le k \le n.$$

Now for any $p = \sum_{j=0}^{n} a_j z^j \in \mathcal{P}_n$

(87.2) $$(\tau_n p)(z) = \sum_{k=0}^{n} a_k \sum_{j=0}^{n} \varphi_{j-k} z^j = \sum_{k=0}^{n} a_k \sum_{j=0}^{n} \int_\Gamma (z')^{k-j} \varphi(z') \frac{dz'}{2\pi i z'} z^j$$

$$= \sum_{k=0}^{n} a_k \int_\Gamma (z')^{k-1} \varphi(z') \left(\sum_{j=0}^{n} (z/z')^j \right) \frac{dz}{2\pi i}$$

$$= \sum_{k=0}^{n} a_k \int_\Gamma (z')^{k-1} \varphi(z') \frac{(z/z')^{n+1} - 1}{z/z' - 1} \frac{dz'}{2\pi i}$$

$$= \int_\Gamma \varphi(z') p(z') \frac{(z/z')^{n+1} - 1}{z - z'} \frac{dz'}{2\pi i}$$

$$= \int p(z') \frac{(z/z')^{n+1} - 1}{z - z'} \frac{dz'}{2\pi i} - \int K_n(z, z') p(z') dz'$$

where

(88.1) $$K_n(z,z') = \frac{(z/z')^{n+1} - 1}{z - z'} \frac{1 - \varphi(z')}{2\pi i}$$

But $q(z') = \dfrac{p(z') - p(z)}{z' - z}$ is a poly. in z' of degree $\le n-1$.

Hence
$$\int_{\Gamma} q(z') \left((z/z')^{n+1} - 1 \right) dz' = 0.$$

Thus

$$T_n p(z) = p(z) \int \frac{(z/z')^{n+1} - 1}{z - z'} \frac{dz'}{2\pi i} - \int K_n(z,z') p(z') dz'$$

$$= p(z) - \int K_n(z,z') p(z') dz'$$

ie

(88.2) $$T_n p = (I - K_n) p \quad , \quad p \in \mathcal{P}_n$$

Clearly K_n is an integrable operator of the form

$$K_n(z,z') = \frac{f_1(z) g_1(z') + f_2(z) g_2(z')}{z - z'}$$

(88.3) where $f = (f_1, f_2)^T = (z^{n+1}, 1)^T$
$g = (g_1, g_2)^T = \left(z^{-(n+1)} \frac{1 - \varphi(z)}{2\pi i}, - \frac{1 - \varphi(z)}{2\pi i} \right)^T$.

Note that $\langle f(z), g(z) \rangle = 0$ on $\mathbb{T}$.

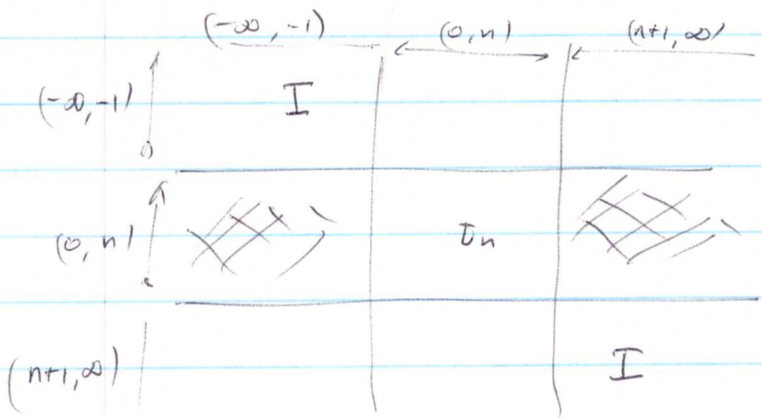
Now from (87.1) and (88.2).

(89.1) $$(1 - K_n) z^k = \sum_{j=0}^{n} \varphi_{j-k} z^j, \qquad 0 \le k \le n.$$

On the other a direct calculation (exercise) shows that for $k < 0$ or $k > n$

(89.2) $$(1 - K_n) z^k = z^k + \sum_{j=0}^{n} \varphi_{j-k} z^j$$

It follows that K_n is finite rank, and hence trace class in $L^2(\mathbb{T})$ and in the orthonormal basis $\{z^k\}_{-\infty}^{\infty}$ for $L^2(\mathbb{T})$, $1 - K_n$ has block form



and so (exercise)

(89.3) $$D_n = \det T_n = \det T_n = \det(1 - K_n)$$

Thus we see that a Toeplitz determinant can always be expressed as a Fredholm determinant of an integrable operator.

The RHP associated with K_n on Γ has jump matrix

(90.0) $$v_n = I - 2\pi i\, f g^T$$

$$= I - 2\pi i \begin{pmatrix} z^{n+1} \\ 1 \end{pmatrix} \begin{pmatrix} z^{-(n+1)} & -1 \end{pmatrix} \frac{1-\varphi}{2\pi i}$$

$$= I - \begin{pmatrix} 1 & -z^{n+1} \\ z^{-(n+1)} & -1 \end{pmatrix} (1-\varphi)$$

$$= \begin{pmatrix} \varphi & -(\varphi-1)\, z^{n+1} \\ z^{-(n+1)}(\varphi-1) & 2-\varphi \end{pmatrix}, \quad z \in \Gamma.$$

To compute D_n we use the elementary formula

(90.1) $$\log D_n = \log \det(1-K_n) = \operatorname{tr} \log(1-K_n)$$

$$= \int_0^1 \frac{d}{dt} \operatorname{tr} \log(1-tK_n).$$

$$= -\int_0^1 \operatorname{tr}\left(\frac{1}{1-tK_n} K_n\right) dt .$$

For $0 \le t \le 1$, set

(91.1) $$\varphi_t(z) = (1-t) + t\,\varphi(z) , \quad z \in T .$$

Clearly $\varphi_t(z) > 0$ and $\varphi_0 = 1$, $\varphi_1 = \varphi$. Now

$\varphi_t - 1 = t(\varphi - 1)$ and so we see from (88.1)

$$tK_n = K_{t,n} = \frac{(z/z')^{n+1} - 1}{z - z'} \, \frac{1 - \varphi_t(z')}{2\pi i}$$

Now it follows from (83.2) that

$$\frac{1}{1-tK_n} tK_n = \frac{1}{1-K_{t,n}} K_{t,n} = \frac{1}{1-K_{t,n}} - 1$$

$$= R_{t,n} = \frac{\sum_{j=1}^{2} F_{t,j}(z) \, G_{t,j}(z')}{z - z'}$$

Hence (exercise).

(91.2) $$\log D_n = -\int_0^1 \operatorname{tr} R_{t,n} \, \frac{dt}{t}$$

$$= -\int_0^1 \left(\int_\Gamma \left(\sum_{j=1}^{2} F'_{t,j}(z) \, G_{t,j}(z) \right) dz \right) \frac{dt}{t}$$

Here

(92.1) $$(F_{t,1}, F_{t,2})^T = m_{t+} f_t = m_{t+} (z^{n+1}, 1)^T$$

$$(G_{t,1}, G_{t,2})^T = (m_{t+}^T)^{-1} \left(z^{-(n+1)} \frac{1-\varphi_t}{2\pi i}, -\frac{1-\varphi_t}{2\pi i} \right)^T$$

and $m_{t\pm}$ solves the normalized RHP $(\Gamma, v_{t,n})$ where $v_{t,n}$ is given by (90.0) with $\varphi(z)$ replaced by φ_t. We will return to the analysis of this problem later on.

<u>Exple 2</u> (Gap probabilities in Random Matrix Theory)

For GUE let $P_N(\alpha, \beta)$ be the probability ("gap prob") that there are no eigenvalues in the interval (α, β). Then for an appropriate scaling γ_N

$$P(a,b) = \lim_{N \to \infty} P_N(\gamma_N a, \gamma_N b) \quad \text{exists}$$

and equals

$$\det(1 - K_{(a,b)})$$

where $K_{(a,b)}$ is the (trace class) operator with

with kernel

$$K_{(a,b)}(z,z') = \frac{\sin \pi(z-z')}{\pi(z-z')}$$

acting on $L^2(a,b)$. Of particular interest is the limiting behavior of

$$P_x = P\left(\frac{xa}{\pi}, \frac{xb}{\pi}\right)$$

as $x \to \infty$. A simple change of variable shows that

$$P_x = \det(I - K_x)$$

where $K_x(z,z') = \dfrac{\sin x(z-z')}{\pi(z-z')}$ acts on the fixed space $L^2(a,b)$. Clearly K_x is of integrable type

$$\frac{f_1(z)g_1(z') + f_2(z)g_2(z')}{z-z'}, \text{ where}$$

$$f^T = (f_1, f_2) = (e^{izx}, e^{-izx})$$

$$g^T = (g_1, g_2) = \left(\frac{e^{-izx}}{2\pi i}, -\frac{e^{izx}}{2\pi i}\right)$$

The jump matrix v_x for the associated RHP $(\Sigma = (a,b), v_x)$

has the form

$$v_x = I - 2\pi i \begin{pmatrix} e^{i\zeta x} \\ e^{-i\zeta x} \end{pmatrix} \left(\frac{e^{-i\zeta x}}{2\pi i} \quad \frac{-e^{i\zeta x}}{2\pi i} \right).$$

$$= \begin{pmatrix} 0 & e^{2i\zeta x} \\ -e^{-2i\zeta x} & 2 \end{pmatrix}$$

A direct calculation (exercise) shows that

$$\frac{d}{dx} \log \det(1 - K_x) = i \left((m_1(x))_{22} - (m_1(x))_{11} \right)$$

where

$$m(\zeta, x) = I + \frac{m_1(x)}{\zeta} + O\left(\frac{1}{\zeta^2}\right), \quad \zeta \to \infty$$

is the solution of the normalized RHP $(\Sigma = (a,b), v_x)$

To determine the behavior of P_x as $x \to \infty$, we see that we must analyze the RHP (Σ, v_x) as $x \to \infty$.

Exple 3   XY transverse Ising spin chain at the critical magnetic field

see [D]

$$H = -\frac{1}{2} \sum_{\ell \in \mathbb{Z}} \left(\sigma_\ell^x \sigma_{\ell+1}^x + \sigma_\ell^z \right).$$

Then (see [D])

$$\chi(t) = \text{autocorrelation} = \langle \sigma_0^x(t)\, \sigma_0 \rangle_T$$

$$= \frac{\operatorname{tr}\left(e^{-\beta H} \left(e^{-iHt} \sigma_0^x e^{iHt} \right) \sigma_0^x \right)}{\operatorname{tr} e^{-\beta H}} \quad , \beta = \frac{1}{T}$$

$$\overset{!}{=} e^{-t^2/2} \det\left(1 - K_t\right)$$

where

$$K_t(z', z') = \varphi(z) \frac{\sin it (z - z')}{\pi (z - z')} \quad , \quad -1 < z < 1$$

and

$$\varphi(z) = \tanh\left(\beta \sqrt{1 - z^2}\right) \quad , \quad -1 < z < 1$$

Clearly K_t is an integrable operator. Basic question:

$$K_t \to ? \quad \text{as } t \to \infty ,$$

Another way Riemann - Hilbert techniques enter analysis, is through the <u>Wiener - Hopf method</u> which came to the fore in the 1930's.

Consider the system of equations in $(L^2(0,\infty))^n$

$$(96.1)\qquad f(x) - \int_0^\infty k(x-y)\, f(y)\, dy = g(x), \quad x > 0$$

Here $k = (k_{ij}(u))_{i,j=1}^n \in (L^1(\mathbb{R}, du))^{n^2}$; $g \in (L^2(0,\infty))^n$

is given and we must solve (96.1) for $f \in (L^2(0,\infty))^n$.

Set

$$F(x) \equiv f(x) \text{ for } x > 0 \text{ and } F(x) \equiv 0 \quad \text{for } x < 0,$$

$$G(x) \equiv g(x) \qquad x > 0$$

$$= -\int_0^\infty k(x-y)\, f(x)\, dx = -\int_{-\infty}^\infty k(x-y)\, F(y)\, dy$$

for $x < 0$

Clearly $F, G \in L^2(\mathbb{R})$.

Then (96.1) takes the form

$$(96.2)\qquad F(x) - \int_{-\infty}^\infty k(x-y)\, F(y)\, dy = G(x), \quad x \in \mathbb{R}.$$

Define the inverse Fourier transforms

$$F(z) \equiv \int_{\mathbb{R}} e^{+izx}\, F(x)\, \frac{dx}{\sqrt{2\pi}}$$

$$G(z) \equiv \int_{\mathbb{R}} e^{+izx}\, G(x)\, \frac{dx}{\sqrt{2\pi}}$$

$$K(z) \equiv \int_{\mathbb{R}} e^{izx}\, k(x)\, dx$$

Taking the inverse transform of (96.2) we