Lecture 7

obtain

(97.1) $$v(z)\, F(z) = G(z) \qquad z \in \mathbb{R}.$$

where

(97.2) $$v(z) = 1 - K(z)$$

Now write
$$G(z) = G_+(z) + G_-(z)$$

where
$$G_+(z) = \int_0^\infty e^{izx} G(x) \frac{dx}{\sqrt{2\pi}} = \int_0^\infty e^{izx} g(x) \frac{dx}{\sqrt{2\pi}}$$
$$G_-(z) = \int_{-\infty}^0 e^{+izx} G(x) \frac{dx}{\sqrt{2\pi}}.$$

With this notation (97.1) takes the form

(97.3) $$v(z) F(z) = G_+(z) + G_-(z).$$

Assuming that

(97.4) $$\det v(z) = \det (I - K(z)) \neq 0, \qquad z \in \mathbb{R}$$

then (97.3) takes the form

$$F(z) = v^{-1}(z)\, G_+(z) + v^{-1}(z)\, G_-(z)$$

or taking transposes,

(97.5) $$m_+(z) = m_-(z)\, V(z) + H(z)$$

where
$$V(z) = (v^{-1}(z))^T$$
$$m_+(z) = F(z)^T$$
$$m_-(z) = G_-(z)^T$$
$$H(z) = (v^{-1}(z)\ G_+(z))^T$$

As $h \in L^1$, $v(z) \to I$ as $z \to \infty$: Hence

$v^{-1} \in L^\infty(\mathbb{R})$, by (97.4), which $\Rightarrow$ H is a

known function in $L^2(\mathbb{R})$.

Now it is an important <u>exercise</u> to show that

$$h = C^+\varphi \text{ for some } \varphi \in L^2(\mathbb{R}) \Leftrightarrow h(z) = \int_0^\infty e^{izx}\psi(x)\frac{dx}{\sqrt{2\pi}}$$

for some $\psi \in L^2(0,\infty)$.

(How are ψ and φ related?)

Similarly
$$h = C^-\varphi \text{ for some } \varphi \in L^2(\mathbb{R}) \Leftrightarrow h(z) = \int_{-\infty}^0 e^{izx}\psi(x)\frac{dx}{\sqrt{2\pi}}$$

for some $\psi \in L^2(0,\infty)$.

It follows that for $\varphi_1, \varphi_2 \in L^2$

$$m_+ = C^+\varphi_1$$
$$m_- = C^-\varphi_2$$

Now as $(C^+)^2 = C^+$, $(C^-)^2 = -C^-$

we see that

$$m_+ = C^+\varphi_+ \quad , \quad \varphi_+ = C^+\varphi,$$
$$m_- = C^-\varphi_- \quad , \quad \varphi_- = -C^-\varphi.$$

But $C^+C^- = C^-C^+ = 0$ and so

(99.1) $$m_\pm = C^\pm \varphi \in \partial C(L^2).$$

where $\varphi = \varphi_+ + \varphi_- \in L^2$.

Thus we are led to a $IRHP2_{L^2}$!

The <u>Wiener-Hopf method</u> is a method to solve IRHP2's.

The idea is to introduce the solution of the normalized RHP $(\mathbb{R}, V)$

(99.2) $$\begin{cases} M_+ = M_- V \\ M_\pm \in I + \partial C(L^2). \end{cases}$$

Assuming that $M_\pm$ $\exists$ <u>and</u> that $M_\pm$ are invertible with

(100.1) $$M_\pm^{-1} \in I + \partial C(L^2).$$

we have

$$m_+ = m_- V + H$$

$$= m_- M_-^{-1} M_+ + H$$

$$\Rightarrow \quad m_+ M_+^{-1} = m_- M_-^{-1} + H M_+^{-1}.$$

This is now an <u>additive</u> RHP which we can solve by the Plemelj formula

$$m \cdot M^{-1} = I + \int_{\mathbb{R}} \frac{H M_+^{-1}(s)}{s-z} \frac{ds}{2\pi i}$$

or

(100.2) $$m_+(z) = M_+(z) + \int_{\mathbb{R}} \frac{H(s)\, M_+^{-1}(s)\, M_+(z)}{s - z_+} \frac{ds}{2\pi i}$$

$$= \left[I + \left(C^+ H M_+^{-1} \right) \right] M_+$$

The method works if we know that the map

$$H \mapsto (C^+ H m_+^{-1}) m_+ .$$

maps L^2 to L^2. This would be true in particular __if__ we know that the normalized RHP $(\mathbb{R}, V)$ had a classical solution so that m_+, m_+^{-1} are continuous and bded on $\mathbb{R}$.

Once we have $m_+(z) = F(z)^T$, we immediately have the desired solution

$$f(x) = F(x) = \int_{-\infty}^{\infty} e^{-izx} m_+^T(z) \frac{dz}{\sqrt{2\pi}}, \qquad x > 0 .$$

__Remark__

In our earlier calculations we related the solution of an $IRHP2_{L^2}$ to the invertibility of the operator $(1 - C_\omega)$. Here we only seem to need to consider the solution of $(1 - C_\omega)\mu = I$ i the $IRHP1_{L^2}$ for a particular RHS namely I. We will clarify the situation later on.

We now consider another more classical source of RHP's (Ref: D + Zhou (CPAM 56 (2003) 1029 - 1077)

Consider the ZS-AKNS operator (Zakharov-Shabat: Ablowitz-Kaup-Newell-Segur) for the defocussing Non Linear Schrödinger equation (NLS):

(102.1) $$\left[\partial_x - \left(iz\sigma + \begin{pmatrix} 0 & q(x) \\ \bar{q}(x) & 0 \end{pmatrix} \right) \right] \psi = 0$$

Here $z \in \mathbb{C}$, $\sigma = \frac{1}{2}\sigma_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Here $q(x) \to 0$ fast enough as $|x| \to \infty$: at least $q \in L^1(\mathbb{R})$. As noted earlier, equation (102.1) is intimately connected with the defocusing NLS equation by virtue of the fact that the operator (cf Eyple 1 ankdU p7)

$$L = (i\sigma)^{-1}\left(\partial_x - \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}\right)$$

undergoes an <u>isospectral deformation</u> if $q = q(t) = q(x,t)$ solves NLS,

(103.1)
$$iq_t + q_{xx} - 2|q|^2 q = 0$$
$$q(x, t=0) = q_0(x)$$

To see this we note that the equations (the <u>Lax Pair</u> for NLS)

(103.2)
$$\partial_x \psi = P\psi$$
$$\partial_t \psi = Q\psi$$

where

(103.3)
$$P = iz\sigma + Q_1, \qquad Q_1 = \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}, \quad z \in \mathbb{C}$$
$$Q = -iz^2\sigma - zQ_1 + Q_2, \qquad Q_2 = \begin{pmatrix} -i|q|^2 & iq_x \\ -i\bar{q}_x & i|q|^2 \end{pmatrix}$$

are compatible $\Leftrightarrow$ $q = q(x,t)$ solves NLS

Indeed suppose that ψ is a fundamental solution of (103.2) i.e. $\det\psi \neq 0$.

Then, $(\psi_x)_t = (P\psi)_t = P_t\psi + P\psi_t = P_t\psi + PQ\psi$

$$(\psi_t)_x = (Q\psi)_x = Q_x\psi + Q\psi_x = Q_x\psi + QP\psi$$

and equating cross-derivatives we find

$$(104.1) \qquad P_t = Q_x + [Q, P] = Q_x + (QP - PQ)$$

Thus

$$P_t = \begin{pmatrix} 0 & q_t \\ \bar{q}_t & 0 \end{pmatrix} = -z \begin{pmatrix} 0 & q_x \\ \bar{q}_x & 0 \end{pmatrix} + \begin{pmatrix} -i|q|^2_x & iq_{xx} \\ -i\bar{q}_{xx} & +i|q|^2_x \end{pmatrix}$$

$$\cancel{+ (-iz^2)[\sigma, Q_1]}$$

$$\cancel{+ (-z)(iz)[Q_1, \sigma]}$$

$$+ iz[Q_2, \sigma]$$

$$+ [Q_2, Q_1].$$

But

$$iz[Q_2, \sigma] = \begin{pmatrix} 0 & z q_x \\ z\bar{q}_x & 0 \end{pmatrix}$$

and

$$[Q_2, Q_1] = \begin{pmatrix} i|q|^2_x & -2iq|q|^2 \\ 2i\bar{q}|q|^2 & -i|q|^2_x \end{pmatrix}$$

Hence

$$\begin{pmatrix} 0 & q_t \\ \bar{q}_t & 0 \end{pmatrix} = \begin{pmatrix} -i|q|^2_x + i|q|^2_x & \cancel{-zq_x} + iq_{xx} \cancel{+zq_x} - 2iq|q|^2 \\ \cancel{-z\bar{q}_x} - i\bar{q}_{xx} \cancel{+z\bar{q}_x} + 2i\bar{q}|q|^2 & i|q|^2_x - i|q|^2_x \end{pmatrix}$$

$$= \begin{pmatrix} 0 & iq_{xx} - 2iq|q|^2 \\ -i\bar{q}_{xx} + 2i\bar{q}|q|^2 & 0 \end{pmatrix}$$

which is just the NLS eqtn (103.1). (103.2)

<u>Exercise</u>: Show that if q satisfies NLS, then $\exists$ fundamental solution ψ of

To see that $L = L(q(t))$ undergoes an iso-spectral deformation if $q(t)$ solves NLS, consider for definiteness the spatially periodic case $q(x,t) = q(x+1,t)$ and let z be an eigenvalue of $L = L(t)$ ie

$$L\psi = z\psi \quad , \quad \psi(x+1,t) = \psi(x,t).$$

Now note that (104.1) can be written in "iso-spectral" form

(105.1) $$(\partial_x - P)_t = [Q, \partial_x - P]$$

As

$$0 = (L - z)\psi = \frac{1}{i\sigma}(\partial_x - Q_1 - iz\sigma)\psi = \frac{1}{i\sigma}(\partial_x - P)\psi$$

we have $(\partial_x - P)\psi = 0$ and so (recall z = const in (105.1)).

$$\begin{aligned} 0 &= (\partial_x - P)_t \psi + (\partial_x - P)\psi_t \\ &= Q(\partial_x - P)\psi - (\partial_x - P)Q\psi - i\dot{z}\sigma\psi \\ &\quad + (\partial_x - P)\psi_t \\ &= (\partial_x - P)(\psi_t - Q\psi) - i\dot{z}\sigma\psi \end{aligned}$$

$\dot{z} = \frac{\partial z}{\partial t}$

is

$$(\partial_x - P)(\psi_t - Q\psi) = i\dot{\zeta}\psi$$

or

$$(L-\zeta)(\psi_t - Q\psi) = \dot{\zeta}\psi$$

Now as $L = L^*$ and so $\zeta = \zeta(t)$ is real we can take inner products to obtain

$$\dot{\zeta}\int_0^1 |\psi|^2 = \int_0^1 \bar{\psi}(L-\zeta)(\psi_t - Q\psi)$$

$$= \int_0^1 \overline{(L-\zeta)\psi}\,(\psi_t - Q\psi)$$

$$= 0$$

and so $\dot{\zeta} = 0$ $\Rightarrow$ <u>iso-spectral</u> <u>deformation</u> of L.

So how should we interpret "iso-spectral deformation" in the whole line case $-\infty < x < \infty$? Here there should be continuous spectrum for L in addition to any possible $L^2(\mathbb{R})$ eigenvalues.

What "stays constant" under NLS?

The key is to focus on the eigenfunctions, not the eigenvalues. In place of periodic solutions ψ of $L\psi = \lambda\psi$, $\psi(x+1) = \psi(x)$, we must seek <u>distinguished</u> solutions of the equation $\partial_x \psi = P\psi$.

Note the following: suppose $q(x)$ has compact support, say $q(x) = 0$ if $|x| > L$. Then

(107.1) $$\partial_x \psi = P\psi = (iz\sigma + Q_1)\psi = iz\sigma\psi, \quad |x| > L$$

and so

$$\psi = e^{iz\sigma} A_\pm, \quad x > L, \; x < -L \text{ resp.}$$

In particular, for $\operatorname{Im} z > 0$,

$$\psi_+ = e^{izx/2} e_1 \qquad e_1 = \begin{pmatrix}1\\0\end{pmatrix}$$
$$\psi_- = e^{-izx/2} e_2 \qquad e_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

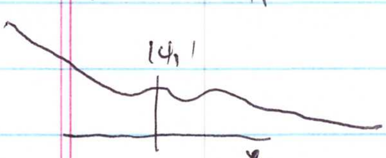
are 2 solutions of (107.1) in $|x| > L$ which decay at $x = \pm\infty$ resp., and grow exponentially at $x = \mp\infty$

Let $\psi_1(x,z)$ be the solution of (107.1) which equals $\psi_+(x,z)$ for $x > L$. Then for $x < -L$,

$$\psi_1(x,z) = a_1\,\psi_+(x,z) + b_1\,\psi_-(x,z)$$

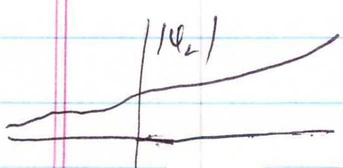
for suitable constants $a_1 = a_1(z)$, $b_1 = b_1(z)$. Now $a_1(z) \neq 0$: for if $a_1(z) = 0$, we would then have an exponentially decaying, and hence $L^2(\mathbb{R})$ solution of $(L - z)\psi = 0$. As L is self-adjoint, and $z \notin \mathbb{R}$, this is impossible.

Hence $a_1 \neq 0$ and we see that



$\psi_1(x,z)$ decays exponentially as $x \to +\infty$
grows exponentially as $x \to -\infty$.

Similarly we see that



$\psi_2(x,z)$ decays exponentially as $x \to -\infty$.
grows exponentially as $x \to +\infty$