Lecture 8

Said differently, we have produced a solution

$$\psi(x,z) = \left( \frac{\psi_1(x,z)}{a_1(z)}, \psi_2(x,z) \right), \quad \operatorname{Im} z > 0$$

of (107.1) st

(109.1) $$\begin{cases} \psi(x,z) e^{-ixz\sigma} \to I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{as } x \to -\infty \\ \text{and is bounded as } x \to +\infty. \end{cases}$$

Such solutions are called <u>Beals-Coifman (BC) solutions</u> of (107.1)

Such a solution of (107.1), even in the general case $q \in L^1(\mathbb{R})$, if it existed, would be unique.

Indeed, suppose there were 2 such solutions, $\psi$ and $\tilde{\psi}$. Then as $\operatorname{tr}(iz\sigma + Q_1) = 0$ it follows that $\det \psi(x,z) = \text{const} = c(z)$

But as $x \to \infty$, $\det \psi(x,z) = \det(\psi(x,z) e^{-ixz\sigma}) \to 1$ and so $c(z) = 1$. Hence, $\exists$ matrix $B = B(z)$ such that $\tilde{\psi}(x,z) = \psi(x,z) B$.

Write $\tilde{\psi} = \tilde{m} e^{iz x\sigma}$, $\psi = m e^{iz x \sigma}$, where

$\tilde{m}, m \to I$ as $x \to -\infty$
and are bded as $x \to +\infty$.

Then

$$\tilde{m} = m \; e^{izx\sigma} \; B \; e^{-izx\sigma}$$
$$= m \begin{pmatrix} B_{11} & B_{12} e^{izx} \\ B_{21} e^{-izx} & B_{22} \end{pmatrix} \qquad (110.1)$$

Letting $x \to -\infty$, we conclude that $B_{12} = 0$ and

$B_{11} = B_{22} = 1$ as $\tilde{m}, m \to I$. But then

letting $x \to +\infty$, we conclude that $B_{21} = 0$. Hence

$B = I$ and so $\psi = \tilde{\psi}$.

We also have uniqueness for Beals-Coifman

solutions in $\operatorname{Im} z < 0$.

(Note that such B-C solutions do not exist for

$z \in \mathbb{R}$: we needed the exponential growth in (110.1)

to conclude uniqueness).

Now there is a mantra in the theory of integrable systems, and indeed in mathematics at large, that if an object is singled out, or uniquely specified, then it's the <u>right</u> object to consider. The converse of this statement is probably even more true: if you cannot define precisely what one means by something, then you probably won't get very far!

So one concentrates on these BC solutions. It turns out (see [D Zhou] above) that such solutions $\psi(x,z)$

- exist for all $q \in L^1$, $z \notin \mathbb{R}$.
- are analytic in $\mathbb{C} \setminus \mathbb{R}$ for any $x \in \mathbb{R}$.
- are continuous down to the axis,

$$\psi_\pm(x,z) = \lim_{\varepsilon \downarrow 0} \psi(x, z \pm i\varepsilon)$$

and satisfy

- $\psi(x,z)\, e^{-ixz\sigma} \to I$ as $z \to \infty$ in $\mathbb{C} \setminus \mathbb{R}$

To higher order, $\psi e^{-izx\sigma} = I + \frac{m_1}{z} + O(\frac{1}{z^2})$ and we have, in particular, $q(x) = -i\,(m_1(x))_{12}$.

Now clearly $\psi_\pm(x,z)$, $z \in \mathbb{R}$, are 2 fundamental solutions ($\det \psi_+ = \det \psi_- \equiv 1$) of (107.1). Hence

$$\psi_+(x,z) = \psi_-(x,z)\, v(z)$$

for some matrix $v(z)$, $\det v(z) = 1$. Direct calculation (see [DZ]) shows that $v(z)$ is of the form

(112.1)
$$v(z) = \begin{pmatrix} 1 - |r(z)|^2 & r(z) \\ -\overline{r(z)} & 1 \end{pmatrix}.$$

where

$$r = r(z) = \text{"reflection coefficient"}$$

(112.2)
$$\|r\|_{L^\infty} < 1$$

To summarize we see that

(112.3)
$$m = m(x,z) = \psi(x,z)\, e^{-ixz\sigma}$$

is a <u>classical</u> <u>solution</u> of the normalized RHP

(113.1) $\left(\Sigma = \mathbb{R},\ v_x = e^{iz x\sigma} v\, e^{-iz x \sigma} = \begin{pmatrix} 1-|r|^2 & r e^{izx} \\ -\bar{r} e^{-izx} & 1 \end{pmatrix}\right)$

In this way the scattering problem (107.1) gives rise to a RHP. Scattering and inverse scattering theory consists of the study of the map

$$q \to r = \mathcal{R}(q)$$

and its inverse $r \mapsto q = \mathcal{R}^{-1}(r)$

$\mathcal{R}$ is constructed as follows from the scatt. prob (107.1)

$$q \mapsto \underset{\substack{\uparrow \\ \text{BC solution}}}{\psi(x,z;q)} \mapsto v_x(z) \longmapsto r \equiv \mathcal{R}(q).$$

$\mathcal{R}^{-1}$ is constructed as follows ~~from~~ the RHP $(\Sigma, v_x)$:

$$r \mapsto v_x \to \underset{\substack{\uparrow \\ \text{normalized} \\ \text{solution of} \\ \text{RHP}}}{m(x,z;r)} = I + \frac{m_1(x,r)}{z} + O\left(\frac{1}{z^2}\right),\ z \to \infty \to -i\left(m_1(x,r)\right)_{12} \equiv q = \mathcal{R}^{-1}(r).$$

At the technical level, one shows (see [DZ]) that $\mathcal{R}$ is a bi-Lipschitz (i.e. $\mathcal{R}$ & $\mathcal{R}^{-1}$ are Lipschitz) isomorphism from

$$H^{1,1} = \{q \in L^2 : xq, q' \in L^2\}$$

onto

$$H_1^{1,1} = H^{1,1} \cap \{\|r\|_\infty < 1\} = \{r \in L^2 : r', zr \in L^2, \|r_\infty\| < 1\}$$

Moreover if $q = q(t)$ solves NLS, $q(x, t=0) = q_0(x) \in H^{1,1}$, then $r(t) = \mathcal{R}(q(t))$ evolves simply (this is the analog for continuous spectrum of the fact that the pt. spectrum, in the periodic case, remains fixed in time)

$$r(t) = r(t, z) = r(t=0, z)\, e^{-itz^2}, \quad z \in \mathbb{R}.$$

Thus we have the <u>solution procedure</u> for NLS:

$$(114.1) \qquad q(t) = \mathcal{R}^{-1}\left(e^{-it\diamond^2} \mathcal{R}(q)\right)$$

The efficacy of the method to compute the long time asymptotics of $q(t)$, rests on the fact that we can control the solution of the oscillatory RHP $\left(\Sigma,\ v_{x,t} = \begin{pmatrix} 1-|r|^2 & r e^{i\theta} \\ -\bar{r} e^{-i\theta} & 1 \end{pmatrix}\right)$

for $\theta = xz - tz^2$, as $t \to \infty$, $x \in \mathbb{R}$, using

the non-commutative steepest-descent method for RHP's.

Think of $\mathcal{R}$ as a non-linear version of the Fourier transform $\mathcal{F}$ & indeed, one can show that for $r$ small, $r(z) = \mathcal{R}(q)(z) \sim \int q(y) e^{-izy}\,dy \approx (\mathcal{F}q)(z)$.

Thus we have seen 4 sources for RHP's:

1) integrable operators

2) Wiener-Hopf type problems: $(1-K)f = g$ on $L^2(\mathbb{R}_+)$, $k(x,y) = k(x-y)$.

3) scattering problems

4) problems from "out of the blue" eg RHP for orthog. polynomials ([FIK]).

The Painlevé equations (see [Fokas, Its, Novokshenov, Kapaev]) are also described by

RHP's: such RHP's arise by considering Lax-pairs as in 3/ above. Recall that an equation of the form

(116.1)
$$y'' = F(x, y, y')$$
$$y(x_0) = a, \quad y'(x_0) = b$$

( F meromorphic in x, rational and y, y' )

has the <u>Painlevé property</u> if the following is true: the only singularities in the complex plane of the solution $y = y(x; a, b)$ of (116.1) that are allowed to move with a, b, are poles. Any essential sing's, branch pts, ..., must remain fixed as a, b vary. Up to changes of variables, there are precisely 6 <u>new</u> Painleve eqtns.

the so-called <u>Painlevé transcendents</u>

Ref: "Painlevé Transcendents" A. Fokas, A. Its, A. Kapaev, V. Novokshenov, AMS, 2006

<u>Exercise</u>: Consider:

(i) $y'' = y$

(ii) $y'' = -\frac{1}{4y^3}$

(iii) $y'' = 2yy'$

Show that (i) (iii) have the Painlevé property, but (ii) does not.

The general form of the Painlevé II equation is

(116.2)
$$u'' = xu + 2u^3 + \nu, \quad \nu = \text{const}.$$

We will illustrate the situation for PII with $\nu = 0$ (see [Fok, Its, Kap, Nov] for $\nu \neq 0$ case)

(117.1) $$u'' = xu + 2u^3$$

In particular, we will show how to construct the assoc. RHP & use it prove the Painlevé prop. for (117.1). Similar considerations apply for all 6 Painlevé eqtns (see Its et al). History: Ablowitz + Segur, Sato-Miwa-Jimbo, Flaschka + Newell.

The key fact is that PII is equivalent to the compatibility of the following Lax Pair:

(117.2) $$\frac{\partial \psi}{\partial z} = \begin{pmatrix} -4iz^2 - ix - 2iu^2 & 4iuz - 2w \\ -4uiz - 2w & 4iz^2 + ix + 2iu^2 \end{pmatrix} \psi \equiv L\psi$$

(117.3) $$\frac{\partial \psi}{\partial x} = \begin{pmatrix} -iz & iu \\ -iu & iz \end{pmatrix} \psi \equiv P\psi.$$

Here $u = u(x)$, $w = w(x)$. It is a simple exercise to verify that the compatibility of 2nd derivatives
$$\partial_x \partial_z \psi = \partial_z \partial_x \psi$$

is equivalent to the relations

(118.1) $$w = u_x, \qquad -w_x + xu + 2u^3 = 0$$

and so

$$u_{xx} = xu + 2u^3$$

(118.2) Note that (117.2) (117.3) $\equiv$ $L_x = P_z + [P, L]$ $\equiv$ $[\partial_x - P, \partial_z - L] = 0$

In analogy with NLS, the analysis of PII proceeds by considering distinguished solutions of the spectral problem

(118.3) $$\frac{\partial \psi}{\partial z} = L\psi$$

(Note that in contrast with NLS, $z$ now plays the role of $x$ and $x$ plays the role of $t$). Write

$$L = z^2 A_2 + z A_1 + A_0$$

where

(118.4) $$\begin{cases} A_2 = -4i\,\sigma_3 \\ A_1 = -4u\,\sigma_2 \\ A_0 = -(ix + 2iu^2)\sigma_3 - 2w\sigma_1 \end{cases}$$

($\sigma_i$ are the Pauli matrices,

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

We think of $x$ as fixed and $u(x)$, $w(x) = u'(x)$ ies fixed and seek $\psi = \psi(z; x)$ as a function of $z$

Now there is a general theory for solutions of such eqtns (118.3)(118.4) (see eg. W. Wasow, Asymptotic Expansions for ode's, Interscience, NY, 1965, Chap IV)

The pt $z = \infty$ is a so-called irregular singular pt. for (118.3) which is a particular exple of an equation of the form

(119.1) $$\frac{\partial \psi}{\partial z} = z^q A(z) \psi \qquad , \quad q \in \mathbb{N}_0$$

where, $A(z) = A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \frac{A_3}{z^3} + \cdots$

$A(z)$ has an asymp. expansion

Thm (Wasow p60 thm 12.3; see also Thm 12.2, p58)

Let $S$ be an open sector in the $z$ plane with vertex at the origin and a positive opening angle not exceeding $\pi/q+1$. Let $A(z)$ be an $n \times n$ matrix function holomorphic in $S$ for $|z| \geq z_0 > 0$ and admitting an asymptotic series

(120.1) $$A(z) = A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots, \quad z \to \infty, \ z \in S$$

Assume in addition that the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ of $A_0$ are distinct. Then (119.1) admits a fundamental matrix solution of the form

(120.2) $$Y(z) = \hat{Y}(z)\, z^D\, e^{Q(z)}$$

Here $Q(z)$ is a diagonal matrix whose diagonal elements are polynomials of degree $q+1$. The leading term of $Q(z)$ is

(120.3) $$\frac{z^{q+1}}{q+1} \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

$D$ is a constant diagonal matrix, and the matrix $\hat{Y}(z)$ has an <u>asymptotic</u> expansion as $z \to \infty$, $z \in S$,