Proof: Recall that μ^{eq} has compact support and $H(\mu^{eq}) = E^V < \infty$. Suppose $\tilde{\mu} \in \mathcal{M}_1(\mathbb{R})$ has compact support and $H(\tilde{\mu}) < \infty$. Then writing

$$\mu_t = t\tilde{\mu} + (1-t)\mu^{eq} = \mu^{eq} + t(\tilde{\mu} - \mu^{eq}) \in \mathcal{M}_1(\mathbb{R}), \quad 0 \le t \le 1$$

we have

(152.1)
$$\begin{aligned} H(\mu_t) = &\iint \log|x-y|^{-1} d\mu^{eq}(x)\, d\mu^{eq}(y) + \int V(x)\, d\mu^{eq}(x) \\ &+ 2t \iint \log|x-y|^{-1} d\mu^{eq}(y)\, d(\tilde{\mu} - \mu^{eq})(x) \\ &+ t \int V(x)\, d(\tilde{\mu} - \mu^{eq})(x) \\ &+ t^2 \iint \log|x-y|^{-1} d(\tilde{\mu} - \mu^{eq})(x)\, d(\tilde{\mu} - \mu^{eq})(y). \end{aligned}$$

All these steps are justified as $\mu^{eq}, \tilde{\mu}$ have compact support and $H(\mu^{eq})$, $H(\tilde{\mu}) < \infty$.

As $H(\mu_t) \ge H(\mu^{eq})$, $0 \le t \le 1$, it is clearly necessary that

$$\int \left[2 \int \log|x-y|^{-1} d\mu^{eq}(y) + V(x) \right] d(\tilde{\mu} - \mu^{eq})(x) \ge 0$$

i.e.

(153.1) $$\int \left[2\int \log|x-y|^{-1}\, d\mu^{eq}(y) + V(x) \right] d\tilde{\mu}(x) \geq \ell$$

where

(153.2) $$\ell = \int \left[2\int \log|x-y|^{-1}\, d\mu^{eq}(y) + V(x) \right] d\mu^{eq}(x)$$

This proves (i) ______________________________

Let

$$B = \{x : 2\int \log|x-y|^{-1}\, d\mu^{eq}(y) + V(x) < \ell\}$$

Clearly B is a bounded set as $\frac{V(x)}{\log(x^2+1)} \to +\infty$ as $|x| \to \infty$. Now suppose $\tilde{\mu}(B) > 0$ and set

(153.3) $$\tilde{\mu}_B = \frac{\chi_B}{\tilde{\mu}(B)}\,\tilde{\mu}$$

where χ_B is the characteristic function of the set B. Then $\tilde{\mu}_B \in M_1(\mathbb{R})$ and has compact support and $H(\tilde{\mu}_B) < \infty$ (why?). Inserting $d\tilde{\mu}_B$ into (153.1) we find

$$\ell \leq \int \left[2 \int \log|x-y|^{-1} d\mu^{eq}(y) + V(x) \right] \frac{\chi_B}{\tilde{\mu}(B)} d\tilde{\mu}(x) < \ell$$

which is a contradiction. Hence

$$(154.1) \qquad \tilde{\mu}(B) = \tilde{\mu}\left(\{ x : 2 \int \log|x-y|^{-1} d\mu(y) + V(x) < \ell \} \right) = 0$$

for all measures $\tilde{\mu}$ with compact support and $H(\tilde{\mu}) < \infty$, and in particular for $\tilde{\mu} = \mu^{eq}$.

But from (153.2)

$$0 = \int \left[2 \int \log|x-y|^{-1} d\mu^{eq}(y) + V(x) - \ell \right] d\mu^{eq}(x)$$

$$= \int_{\mathbb{R} \setminus B} \left[2 \int \log|x-y|^{-1} d\mu^{eq} + V(x) - \ell \right] d\mu^{eq}(x).$$

by (154.1), and so (ii) follows

Conversely, suppose $\mu \in M_1(\mathbb{R})$, satisfies (i) + (ii) and, $H(\mu) < \infty$, and μ has compact support. Then write $\mu^{eq} = \mu + (\mu^{eq} - \mu)$. As in (152.1), we find

$$H(\mu) \geq H(\mu^{eq})$$

$$= H(\mu) + \int \left[2 \int \log|x-y|^{-1} d\mu(y) + V(x) \right] d(\mu^{eq} - \mu)(x)$$

$$+ \iint \log|x-y|^{-1} d(\mu^{eq} - \mu)(x)\, d(\mu^{eq} - \mu)(y)$$

By (i) and (ii) the second term reduces to

$$\int \left[2 \int \log |x-y|^{-1} d\mu(y) + V(x) \right] d\mu^{eq} - \ell$$

$$\geq \ell - \ell = 0$$

But the third term is strictly positive (see (126.11)) unless $\mu^{eq} = \mu$. On the other hand, if $\mu^{eq} \neq \mu$ then the above calculation shows that

$$H(\mu) \geq H(\mu^{eq}) > H(\mu),$$

a contradiction. Thus if μ satisfies (i) (ii) above, it must be μ^{eq}. $\square$.

If we know a priori that $d\mu^{eq} = \psi(x)\,dx$ for some (positive) continuous function ψ, say, of compact support, then $2 \int \log |x-y|^{-1} d\mu^{eq}(y) + V(x)$ is continuous (why?) and Thm 151.1 takes the following stronger form:

Th^m 156.1 (Variational eqtns: strong form).

Suppose $q\mu^{eq} = \psi(x)\,dx$ for some cont. (pos.) func. ψ of compact supp. Then (i) (ii) above can be replaced by

(i)′ $\quad 2\int \log|x-y|^{-1} q\mu^{eq}(y) + V(x) \geq \ell \quad \forall x$

(ii)′ $\quad 2\int \log|x-y|^{-1} q\mu^{eq}(y) + V(x) = \ell$ on $\{x : \psi(x) > 0\}$.

We now show how to use Th^m 156.1 to compute μ^{ed} for the case $V(x) = t x^{2m}$, $m \geq 1$, $t > 0$.

We seek μ^{ed} in the form

(156.2) $\quad \mu^{ed} = \psi(x)\,dx \quad$ with $\quad \int \psi(x)\,dx = 1$

for some continuous $\psi(x) \geq 0$ of compact support, which satisfies (i)′, (ii)′ above. If we succeed in producing such a $\psi(x)$, then $\mu = \psi(x)\,dx$ is necessarily μ^{ed} by Th^m 156.1

Now observe that the weak derivative of the function

$$F(x) = 2\int \log|x-y|^{-1}\,\psi(y)\,dy$$

is given by

$$DF(\phi) = -2\int \phi'(x)\int \log|x-y|^{-1}\,\psi(y)\,dy, \quad \phi\in C_0^{\infty}(\mathbb{R})$$

$$= \lim_{\varepsilon\downarrow 0}\int \phi'(x)\left(\int \log(|x-y|^2+\varepsilon^2)\,\psi(y)\,dy\right)$$

(by dominated convergence: indeed as ψ,ϕ have compact support and $\exists\ C>0$ st

$$|\log|x-y|^2+\varepsilon^2| \le |\log|x-y|^2| + \log(C^2+1)$$

$\forall\ 0<\varepsilon\le\varepsilon_0<1$.)

$$= -\lim_{\varepsilon\downarrow 0}\int \phi(x)\int \frac{2(x-y)}{(x-y)^2+\varepsilon^2}\,\psi(y)\,dy$$

$$= -\int \phi(x)\,2\pi\,H\psi(x)\,dx$$

where

(157.1) $$H\psi(x) = \frac{1}{\pi}\,⨍\,\frac{\psi(y)}{x-y}\,dy$$

is the Hilbert transform of ψ. Here we have used the fact that $\psi \in C_c(\mathbb{R}) \subset L^2(\mathbb{R})$ and

$$\frac{1}{\pi} \int \frac{x-y}{(x-y)^2+\varepsilon^2} \psi(y)\, dy \to H\psi(x)$$

in $L^2(\mathbb{R})$ (see eg [Y. Katznelson], An introduction to harmonic analysis)

It follows that F has a distrib. deriv. in $L^2(\mathbb{R})$ and from (ii)' we must have from (ii)'

(158.1) $$-2\pi H\psi(x) + V'(x) = 0 \qquad \text{a.e. on } \{\psi(x) > 0\}.$$

Define the Borel transform G of ψ by

(158.2) $$G(z) = \frac{1}{i\pi} \int \frac{\psi(y)}{y-z}\, dy \qquad , z \in \mathbb{C} \setminus \operatorname{supp} \psi$$

Note that G is analytic on $\mathbb{C} \setminus \operatorname{supp} \psi$. Now by standard theory (see again [Katznelson]) the limits

(158.3) $$G_\pm(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{i\pi} \int \frac{\psi(y)}{y-(x \pm i\varepsilon)}\, dy$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{i\pi} \int \frac{y-x \pm i\varepsilon}{(y-x)^2+\varepsilon^2} \psi(y)\, dy$$

exists in $L^2(\mathbb{R}, dx)$, and also pointwise a.e., and

(159.1) $$G_\pm(x) = \pm \psi(x) + i H\psi(x)$$

We learn that, a.e. on $\psi(x) > 0$

(159.2) $$G_+(x) + G_-(x) = 2i H\psi(x) = \frac{i}{\pi} V'(x)$$

Combining this with

(159.3) $$G(z) \to 0 \quad \text{as } z \to \infty, \text{ we}$$

see that (159.2) & (159.3) give a scalar Riemann-Hilbert problem (RHP) for G, and hence for ψ. This RHP is not in standard form because of the sum $G_+ + G_-$ in (159.2) rather than the difference. In special cases, however, this can be converted into a standard RHP. Suppose, for example, that

(159.4) $$\left\{ \begin{array}{l} \text{the set } \{\psi(x) > 0\} \text{ consists of a finite \# of (disjoint) intervals} \\ \{\psi(x) > 0\} = \cup_{i=1}^{k} (a_i, b_i) = \Sigma \end{array} \right.$$

$$\overline{a_1 \quad b_1} \quad \overline{a_2 \quad b_2} \quad \cdots \quad \overline{a_k \quad b_k} \qquad \Sigma$$

Let

(160.1) $$q(z) = \prod_{i=1}^{k} (z-a_i)(z-b_i)$$

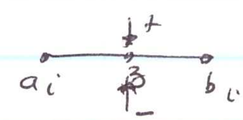
and define $(q(z))^{\frac{1}{2}}$ as an analytic function in $\mathbb{C} \setminus \overline{\Sigma}$ so that

(160.2) $$(q(z))^{\frac{1}{2}} \sim + z^k \quad \text{as } z \to \infty$$

Set

$$\tilde{G}(z) = \frac{G(z)}{(q(z))^{\frac{1}{2}}}$$

Then for $z \in \{\psi(x) > 0\}$ we have



$$\left((q(z))^{\frac{1}{2}}\right)_+ = -\left((q(z))^{\frac{1}{2}}\right)_-$$

Hence for $z \in \{\psi(x) > 0\}$

(160.3) $$\tilde{G}_+(z) - \tilde{G}_-(z) = \frac{G_+(z)}{(q(z)^{\frac{1}{2}})_+} - \frac{G_-(z)}{(q(z))^{\frac{1}{2}}_-}$$

$$= \left[G_+(z) + G_-(z)\right] / (q(z)^{\frac{1}{2}})_+ = \frac{i}{\pi} \frac{V'(z)}{(q(z))^{\frac{1}{2}}_+}$$

On the other hand $(q(z))^{\frac{1}{2}}$ is analytic in $\mathbb{C} \setminus \bar{\Sigma}$ and hence

(161.1) $$\tilde{G}_+(z) - \tilde{G}_-(z) = 0 \quad , \quad z \in \mathbb{R} \setminus \bar{\Sigma}$$

so that $\tilde{G}(z)$ is analytic in $\mathbb{C} \setminus \bar{\Sigma}$

and also, clearly,

(161.2) $$\tilde{G}(z) \to 0 \quad \text{as } z \to \infty$$

Hence we are lead to a standard RHP (160.3) (161.2) on Σ, which can be solved by the Plemelj formula

(161.3) $$\tilde{G}(z) = \frac{1}{2\pi i} \int_\Sigma \frac{\frac{i}{\pi} V'(s)}{(q(s))_+^{\frac{1}{2}}} \frac{ds}{s-z}$$

Indeed, by (159.1) $\tilde{G}_+(z) - \tilde{G}_-(z) = \frac{1}{2}\left[\frac{i}{\pi} V'(z) - \left(-\left(\frac{i}{\pi} V'(z)\right)\right)\right]$ $= \frac{i}{\pi} V'(z)$ for $z \in \Sigma$. And clearly $\tilde{G}(z)$ is analytic in $\mathbb{C} \setminus \bar{\Sigma}$ and $\tilde{G}(z) \to 0$ as $z \to \infty$.

We thus have

(162.1)
$$G(z) = \frac{(q(z))^{\frac{1}{2}}}{2\pi i} \int_\Sigma \frac{\frac{i}{\pi} V'(s)}{(q(s))_+^{1/2}} \frac{ds}{s-z}$$

Now however, in general, $G(z)$ does not decay as $z \to \infty$. Indeed

(162.2)
$$G(z) = \frac{(z^k + \dots)}{2\pi i} \left(-\frac{1}{z}\right) \int_\Sigma ds \frac{\frac{i}{\pi} V'(s)}{(q(s))_+^{\frac{1}{2}}} \left(1 + \frac{s}{z} + \dots + \frac{s^{k-1}}{z^{k-1}} + \dots\right)$$

and to ensure that $G(z) \to 0$ as $z \to \infty$, the

<u>k moment conditions</u>

(162.3)
$$\int_\Sigma \frac{V'(s)}{(q(s))_+^{\frac{1}{2}}} s^j \, ds = 0 \quad , \quad j = 0, \dots, k-1.$$

must be satisfied.

This gives k conditions on the $2k$ endpoints $a_1, b_1, \dots, a_k, b_k$. Furthermore, from (158.2),

$$G(z) = -\frac{1}{i\pi z} \int \psi(y)\, dy + O\left(\frac{1}{z^2}\right)$$

$$= -\frac{1}{i\pi z} + O\left(\frac{1}{z^2}\right)$$

as $\int \psi \, dx = 1$.

This leads to the additional condition from (162.2)

(163.1) $$\frac{i}{2\pi}\int_{\Sigma}\frac{V'(s)\, s^k}{(q(s))_+^{1/2}}\,ds = 1$$

Thus we still require $k-1$ relations to determine the endpoints of Σ. These are obtained via the relation

(163.2) $$\frac{d}{dx}\left[2\int \log|x-y|^{-1}\,\psi(y)\,dy + V(x)\right] = -2\pi H\psi(x) + V'(x)$$

In order that (ii)′ is satisfied with the same constant l in all the k intervals (a_i, b_i), $1 \le i \le k$, we must have

$$\overline{\qquad}_{b_i}\quad{}_{a_{i+1}}\overline{\qquad}$$

(163.3) $$\int_{b_i}^{a_{i+1}}\left(H\psi(u) - \frac{V'(u)}{2\pi}\right)dy = 0 \qquad i = 1, \ldots, k-1$$

Equations (163.3) provide the remaining equations, in addition to (162.3) and (163.1), for $(a_1,b_1), \ldots, (a_k,b_k)$.

In addition to (162.3)(163.1) and (163.3) we have the side conditions

(164.1) $$\operatorname{Re} G_+(x) = \psi(x) \geq 0 \quad \text{and} \quad \{\operatorname{Re} G_+(x) > 0\} = \Sigma$$

i.e. $\operatorname{supp}(\psi(x)\,dx) = \Sigma$.

and (ii)$'$

(164.2) $$2\int \log|x-y|^{-1}\psi(y)\,dy + V(x) \geq \ell \qquad \forall x \in \mathbb{R}.$$

As

$$\ell = 2\int \log|b_i - y|^{-1}\psi(y)\,dy + V(b_i), \quad i = 1, \dots, k-1$$

we see that (164.2) can be written, using (163.2)

(164.3) $$\begin{cases} \displaystyle\int_{b_i}^{x}\left(H\psi(y) - \frac{V'(y)}{2\pi}\right)dy \leq 0, & b_i \leq x \leq a_{i+1}, \quad i = 1, \dots, k-1 \\ \displaystyle\int_{x}^{a_1}\left(H\psi(y) - \frac{V'(y)}{2\pi}\right)dy \geq 0, & x < a_1 \\ \displaystyle\int_{b_k}^{x}\left(H\psi(y) - \frac{V'(y)}{2\pi}\right)dy \leq 0, & x > b_k \end{cases}$$

To summarize: The above calculations show that

if $d\mu^{ed} = \psi(x)\,dx$, where $\psi(x)$ is a cont. function of compact support, $\Sigma = \bigcup_{i=1}^{k}(a_i, b_i)$, then conditions (162.3)(163.1)(163.3)(164.1) and (164.3)

$\Rightarrow$ must be satisfied. Conversely, suppose that Σ is a union of intervals $\bigcup_{i=1}^{k}(a_i, b_i)$, and define $G(z)$ by (162.1) (and suppose that $G(z)$ is contin. down to the axis from $\mathbb{C}_\pm$.) Then if the pair $(\Sigma, \psi(x) \equiv \operatorname{Re} G_+(x))$ satisfies (162.3)(163.1)(163.3)(164.1) and (164.3), then one can show that $\psi(x)dx$ is the equm. meas. for $V(x)$. (Note that as $\psi(x) = \frac{1}{2}(G_+(x) + \overline{G_+(x)})$, $\psi(x)$ is contin. by assumption.)

Indeed, from (162.1), (162.3), we see that $G(z) = O(\frac{1}{z})$ as $z \to \infty$ and hence ~~we are assuming that $G(z)$ is continuous down to the real axis from $\mathbb{C}_+$ and $\mathbb{C}_-$: see below~~, by a simple application of Cauchy's Theorem, we find that

$$G(z) = \frac{1}{2\pi i}\int_{a_1}^{b_k}\frac{(G_+(x) - G_-(x))\,dx}{x - z}$$

However as $(q(s))_+^{\frac{1}{2}} \in i\mathbb{R}$ for $s \in \Sigma$, and $q(z) = \overline{q(\bar{z})}$ for $z \in \mathbb{C}\setminus\Sigma$, we see that

$$G(z) + \overline{G(\bar{z})} = 0 \qquad \forall z \in \mathbb{C}\setminus\Sigma$$

and hence

$$G_+(x) + \overline{G_-(x)} = 0 \qquad \forall x \in \mathbb{R}.$$

and in particular, for $x > b_k$ or $x < a_1$, where $G(z)$ is analytic, $2\psi(x) = G_+(x) + \overline{G_+(x)} = G_+(x) + \overline{G_-(x)} = 0$.
Similarly $\psi(x) = 0$ for x in the gaps, i.e. $x \in (b_i, a_{i+1})$, $i = 1, \dots, k-1$.
(Note: (164.1) could be replaced with the weaker assumption: (164.1)' $\bigcup_{i=1}^{k} (a_i, b_i) \subset \{\psi > 0\}$.
For by the above calculation, we see that $\psi(x) = 0$ on $\mathbb{R} \setminus \Sigma$. Thus (164.1)' $\Rightarrow$ (164.1)).

Thus
$$G(z) = \frac{1}{2\pi i} \int_\Sigma (G_+(s) - G_-(s)) \frac{ds}{s-z}$$

$$= \frac{1}{2\pi i} \int_\Sigma \left(G_+(s) + \overline{G_+(s)}\right) \frac{ds}{s-z}$$

$$= \frac{1}{\pi i} \int_\Sigma \psi(s) \frac{ds}{s-z}.$$

$$= \frac{1}{\pi i} \int \psi(s) \frac{ds}{s-z}.$$

As in (159.1) we learn that $G_\pm(x) = \pm\psi(x) + i H\psi(x)$,

which implies that $G_+ + G_- = 2i H\psi(x)$. But from (162.1),

for $x \in \Sigma$, $G_+(x) + G_-(x) = \frac{i}{\pi} V'(x)$. Hence

$$-2\pi H\psi(x) + V'(x) = 0, \quad x \in \Sigma$$

This implies as before that $2\int \log|x-y|^{-1}\psi(y)\,dy + V(x)$

is constant on $\Sigma = \{\psi(x) > 0\}$, and by (163.3) it must be the same constant, say l, in each interval (a_i, b_i).

Finally, as (164.3) is satisfied by assumption, we conclude that (164.2) is true. It then follows by Th^m 156.1 that $\psi(x)\,dx = \mathrm{Re}\, G_+(x)\,dx$, which is a probability measure with support Σ by (164.1) and (163.1), is the equilibrium measure for $V(x)$.

Important Remark:

As noted above, (161.3) (165.1) and (163.3) give $2k$ conditions for the $2k$ points $a_1, b_1, \ldots, a_k, b_k$. This is true for any $k \geq 1$, and it may happen that we obtain solutions of these equations for many values of k. However, amongst all these solutions only one of them can solve the side conditions (164.1) and (164.3), and that one is the desired solution (see below)

<u>Exercise:</u> Show that if $V \in C^2(\mathbb{R})$, then $G(z)$ defined by (162.1) is continuous up to the boundary from $\mathbb{C}_+$ and $\mathbb{C}_-$