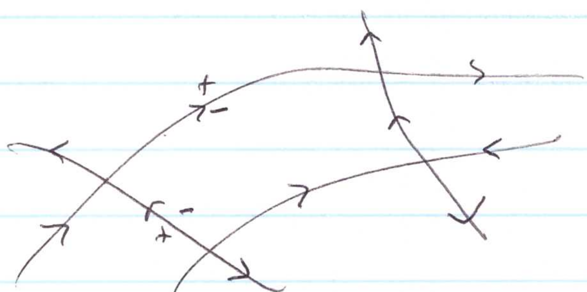
Lecture 12

Let Σ be an oriented contour in the plane



We assume that Σ is a finite union of smooth curves with at most a finite # of pts of self-intersection. By convention, if we move along the contour in the direction of the orientation, we say the (+)-side (resp. (−)-side) lies to the left (resp. right). Let $\mathring{\Sigma} = \Sigma \setminus \{\text{pts of self intersection}\}$.

Let $v : \Sigma \to Gl(k, \mathbb{C})$ be a measurable map with $\|v\|_{L^\infty}$ and $\|v^{-1}\|_{L^\infty}$ finite. Assume in addition that $v(z)$ is continuous in $\mathring{\Sigma}$. We say that an

$m = m(z)$ is a ~~[illegible]~~ matrix valued function from $\mathbb{C} \setminus \Sigma$ to $M(\ell, k; \mathbb{C}) = \{\ell \times k \text{ matrices}\}$

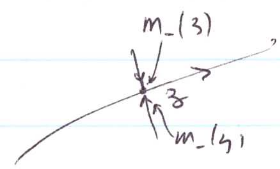
is a <u>solution</u> of the Riemann-Hilbert Problem (RHP) (Σ, v) in the strong sense if

(i) $m(z)$ is analytic in in $\mathbb{C} \setminus \Sigma$ and continuous up to the bdry in each component of $\mathbb{C} \setminus \Sigma$

(ii) For each point $z \in \mathring{\Sigma} \equiv \Sigma \setminus \{$pts. of self-intersection$\}$

$$m_+(z) = m_-(z)\, v(z)$$

where $m_\pm(z) \equiv \lim_{z' \to \pm z} m(z')$



Here $z' \to \pm z$ means $z' \to z$ from the $(\pm)$-side of Σ at z respectively

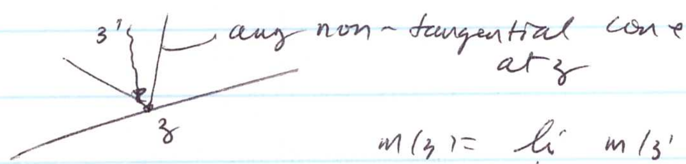
If in addition, $\ell = k$ and

(iii) $m(z) \mapsto I$ as $z \to \infty$ in $\mathbb{C} \setminus \Sigma$
(i.e. given $\varepsilon > 0$ $\exists\, R = R(\varepsilon) > 0$ st $\| m(z) - I \| < \varepsilon$ if $|z| > R$, $z \in \mathbb{C} \setminus \Sigma$)
we say that m is a solution of the

<u>normalized</u> RHP (Σ, v) in the strong sense.

The matrix function v is called the <u>jump matrix</u> for the RHP.
In general, one considers RHP's on for

more general classes of contours and the sense in which $m(z)$ solves the RHP (Σ, v) is generalized: in particular, the requirement of continuity up to the boundary is replaced by the existence of non-tangential limits



$$m(z) = \lim_{z' \to z} m(z').$$

and similarly the normalized condition $m(z) \to I$ as $z \to \infty$, is generalized. (see e.g. P. Deift & X. Zhou CPAM 56 (2003) 1029–1077) For the moment (but see more later), we will only consider RHP's in the strong sense.

We are interested in the RHP for orthogonal polynomials (OP's) ($p_k(x) = \gamma_k x^k + \ldots,\ \gamma_k > 0,\ k = 0, 1, 2, \ldots$) with respect to a (non-trivial) measure $w(x)\,dx$ on $\mathbb{R}$,

(187.1) $$\int p_k(x)\, p_j(x)\, w(x)\,dx = \delta_{kj}, \qquad k, j \geq 0.$$

Such a RHP was discovered by A. Fokas, A. Its and A. Kitaev in the early 90's

Assume that for $j = 0, 1, \dots$

(188.1) $$x^j w(x) \in H^1(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R}) \}$$

This assumption is just for convenience and care of presentation and be weakened considerably.

Note that for any $j \geq 0$

$$\int (1+x^2)^j w(x)\, dx \leq \left(\int (1+x^2)^{-1} dx \right) \left(\int (1+x^2)^{2j+1} w^2(x) dx \right)^{\frac{1}{2}}.$$

$$< \infty$$

Hence all the moments of $w(x)\,dx$ are finite and consequently the p_k's $\exists$, $k \geq 0$. Note also that as $w \in H^1(\mathbb{R})$, $w(x)$ is (absolutely) continous and a simple calculation shows that

$$w^2(x) = -2 \int_x^\infty w(t)\, w'(t)\, dt$$

$$\Rightarrow \quad |w(x)|^2 \leq 2 \left(\int_x^\infty |w|^2 \right)^{\frac{1}{2}} \left(\int_x^\infty |w'|^2 \right)^{\frac{1}{2}}$$

$$\to 0 \text{ as } x \to +\infty$$

Similarly $w(x) \to 0$ as $x \to -\infty$.

Let $\Sigma = \mathbb{R}$ oriented from $-\infty \to +\infty$ $\longrightarrow$ and set

(189.0) $$v(z) = \begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R}$$

Clearly $v(z)$ is cont. on $\mathbb{R}$ and $\|v\|_\infty, \|v^{-1}\|_\infty \in L^\infty(\mathbb{R})$.

For any $n \geq 0$ we seek a 2×2 matrix solution $Y(z) = Y^{(n)}(z) = \{Y^{(n)}_{ij}(z)\}_{i,j=1,2}$ of the (strong) RHP Σ, v i.e.

(191.1)
- $Y(z)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$ and continuous in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$
- $Y_+(z) = Y_-(z)\, v(z)$, $z \in \mathbb{R}$

normalized so that (in the sense of (iii) p186)

- $Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \to I$ as $z \to \infty$ in $\mathbb{C}\setminus\mathbb{R}$

Theorem 189.2 (Fokas-Its-Kitaev) The above normalized RHP has a unique solution (in the strong sense)

(189.3) $$Y(z) = Y^{(n)}(z) = \begin{pmatrix} \pi_n(z) & \int_{\mathbb{R}} \frac{\pi_n(s)\, w(s)}{s-z} \frac{ds}{2\pi i} \\ \mu_{n-1}\, \pi_{n-1}(z) & \int_{\mathbb{R}} \frac{\mu_{n-1}\, \pi_{n-1}(s)\, w(s)}{s-z} \frac{ds}{2\pi i} \end{pmatrix}$$

$n \geq 1$

where

(190.1) $$\pi_n(z) = z^n + a_{n,n-1} z^{n-1} + \cdots + a_{n,0}$$

is the nth monic OP wrt $w(z)$, and

(190.2) $$\mu_{n-1} = -2\pi i \, \gamma_{n-1}^2$$

For $n=0$,

(190.3) $$Y = Y^0(z) = \begin{pmatrix} 1 & \int_\Sigma \frac{w(s)}{s-z} \frac{ds}{2\pi i} \\ 0 & 1 \end{pmatrix}$$

<u>Proof</u>: First we show that if Y with the above properties exists, then Y is unique. Suppose Y is a solution of (191.1) with these prop's.

From (191.1), we have for $z \in \mathbb{R}$,

$$(\det Y)_+(z) = (\det Y_-)(z) \det v(z)$$
$$= (\det Y_-)(z)$$

It follows that $\det Y(z)$ is entire. Moreover

$$\det Y(z) = \det\left(Y(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \right) \to 1 \quad \text{as } z \to \infty$$

(We are using here the simple fact that if

$Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \to I$ as $z \to \infty$ in $\mathbb{C} \setminus \mathbb{R}$, then

$Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \to I$ as $z \to \infty$ in $\mathbb{C}$, as $Y(z)$ is

cont. down to the axis) Hence, by Liouville's thm,

$\det Y(z) \equiv 1$, and so $Y(z)^{-1}$ is analytic in $\mathbb{C} \setminus \mathbb{R}$,

and cont. up to the bdry. Now if $\tilde{Y}(z)$ is a 2nd

solution of the RHP, then across $\mathbb{R} = \Sigma$

$$\begin{aligned}(\tilde{Y} Y^{-1})_+(z) &= \tilde{Y}_+(z) Y_+^{-1}(z) \\ &= \tilde{Y}_-(z)\, v(z)\, (Y_-(z)\, v(z))^{-1} \\ &= (\tilde{Y} Y^{-1})_-(z)\end{aligned}$$

and we conclude as before that $\tilde{Y} Y^{-1}(z)$ is

entire. Moreover

$$(\tilde{Y} Y^{-1})(z) = \left(\tilde{Y}\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix}\right)\left(Y\begin{pmatrix} z^{-n} & 0 \\ 0 & z^{n} \end{pmatrix}\right)^{-1}$$

$$\to I \times I = I \quad \text{as } z \to \infty$$

and again by Liouville's th^m, we conclude

that $\tilde{Y} Y^{-1}(z) \equiv I$ i.e. $\tilde{Y}(z) = Y(z)$.

Suppose that a soln $Y(z)$ of the RHP exists and write

(192.1) $$Y(z) = \begin{pmatrix} Y_{11}(z) & Y_{12}(z) \\ Y_{21}(z) & Y_{22}(z) \end{pmatrix} = \left(I + o(1)\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$$
$$= \begin{pmatrix} z^n + o(z^n) & o(z^{-n}) \\ o(z^n) & z^{-n} + o(z^{-n}) \end{pmatrix}$$

Suppose $n \geq 1$ (The case $n=0$ is left as an exercise).

Consider the 1st row of $Y_+ = Y_- \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$, i.e.

$$\begin{pmatrix} Y_{11} & Y_{12} \end{pmatrix}_+(z) = \begin{pmatrix} Y_{11} & Y_{12} \end{pmatrix}_-(z) \begin{pmatrix} 1 & \omega(z) \\ 0 & 1 \end{pmatrix}$$

In particular $(Y_{11})_+ = (Y_{11})_-$ and therefore $Y_{11}(z)$

is anal. in $\mathbb{C}$ (by Morera's Theorem). But from (192.1) we see

that necessarily $Y_{11}(z)$ is a monic polynomial

$$\hat{\pi}_n(z) = z^n + a_{n,n-1} z^{n-1} + \cdots + a_{n,0}$$

Now

(192.2) $$(Y_{12})_+ = (Y_{12})_- + \hat{\pi}_n \, \omega$$

and from (192.1) $Y_{12}(z) \to 0$ as $z \to \infty$ in $\overline{\mathbb{C}}_+$ & $\overline{\mathbb{C}}_-$.

<u>Claim</u> $Y_{12}(z)$ is given by the <u>Plemelj</u> formula i.e.

(193.1) $$Y_{12}(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\hat{\pi}_n(s)\, w(s)}{s-z}\, ds \quad , \quad z \in \mathbb{C}\setminus\mathbb{R} .$$

To prove (193.1) we use the following basic facts about the <u>Cauchy transform</u> ((non-trivial) exercise)
(cf Lemma 7.12 p.185 Ref #(2))

$$Ch(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{h(s)}{s-z}\, ds \quad , \quad z \in \mathbb{C}\setminus\mathbb{R}$$

for function $h \in H^1(\mathbb{R})$

(193.2) $Ch(z)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$ and continuous in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$

(193.3) $$(C_+h)(z) - (C_-h)(z) = h(z) \quad , \quad z \in \mathbb{R} .$$

(193.4) $$Ch(z) \to 0 \quad \text{as} \quad z \to \infty \quad \text{in} \quad \overline{\mathbb{C}}_+ \cup \overline{\mathbb{C}}_- .$$

Assuming (193.2) – (193.4), and noting that $\hat{\pi}_n(s)\, w(s) \in H^1(\mathbb{R})$ we have that $C(\hat{\pi}_n w)(z)$ is anal. in $\mathbb{C}\setminus\mathbb{R}$ and contin. down to the axis and

$$C_+(\hat{\pi}_n w)(z) = C_-(\hat{\pi}_n w)(z) + \hat{\pi}_n w(z) \quad , \quad z \in \mathbb{R} .$$

It follow that $\Delta(z) = Y_{12}(z) - C(\hat{\pi}_n w)(z)$

is anal. in $\mathbb{C} \setminus \mathbb{R}$, cont. up to the boundary,

and across $\Sigma = \mathbb{R}$, $\Delta_+(z) = \Delta_-(z)$ and so

$\Delta(z)$ is entire, again by Morera's thm. But as $Y_{12}(z) \to 0$ as $z \to \infty$, and

the same is true for $C(\hat{\pi}_n w)(z)$ (193-4), we

conclude that

$$Y_{12}(z) = C(\hat{\pi}_n w)(z)$$

$$= \frac{1}{2\pi i} \int_\Sigma \frac{\hat{\pi}_n(s)\, w(s)\, ds}{s - z}$$

as desired. Using the identity

(194.1) $(s-z)^{-1} = -(z^{-1} + s z^{-2} + \dots + s^{n-1} z^{-n}) + s^{n} z^{-n}(s-z)^{-1}$, we have

$$Y_{12}(z) = \frac{-1}{2\pi i\, z} \int_\Sigma \hat{\pi}_n(s)\, w(s) \left(1 + \frac{s}{z} + \dots + \frac{s^{n-1}}{z^{n-1}} + \frac{s^n}{z^{n-1}} \frac{1}{s-z}\right)$$

As $s^n \hat{\pi}_n(s) w(s) \in H^1$, it follows from (193.4) that $z^{-n} \int_\Sigma \frac{s^n \hat{\pi}_n w\, ds}{s-z}$ $= o(z^{-n})$ and so by (192.1) we must have

$$\int \hat{\pi}_n(s)\, s^j\, w(s)\, ds = 0, \quad j = 0, 1, \dots, n-1.$$

In other words $Y_{11} = \hat{\pi}_n$ must be the n^{th} monic OP associated with $w(x)\,dx$.

From the 2nd row of $Y_+ = Y_- v$, we find as above that

$Y_{21}(z)$ is a polynomial of degree $\leq n-1$

We have also

$$(Y_{22})_+(z) = (Y_{22})_-(z) + Y_{21}\, w(z)$$

and again as $Y_{22}(z) = O(z^{-n}) \to 0$ as $n \to \infty$, $Y_{22}(z)$ is given by the Plemelj formula and (194.1)

$$Y_{22}(z) = \frac{1}{2\pi i}\int_\Sigma \frac{Y_{21}(s)\, w(s)}{s-z}\, ds$$

$$= -\frac{1}{2\pi i z}\int_\Sigma \left(Y_{21}(s)\, w(s)\right)\left(1 + \frac{s}{z} + \cdots + \frac{s^{n-1}}{z^{n-1}} + \frac{s^n}{z^{n-1}} \cdot \frac{1}{s-z}\right)$$

But $Y_{22}(z) = z^{-n} + o(z^{-n})$, and so arguing as on p194, we must have $z^{-n}\int_\Sigma s^n\, Y_{21}\, w\, ds/_{s-z} = o(z^{-n})$ and hence.

(195.1) $$\int_{\mathbb{R}} Y_{21}(s)\, w(s)\, s^j\, ds = 0 \qquad 0 \leq j \leq n-2.$$

and

(196.1) $$\frac{-1}{2\pi i}\int Y_{21}(s)\,w(s)\,s^{n-1}\,ds = 1.$$

Let

$$h(z) = -2\pi i\,\gamma_{n-1}^2\,\pi_{n-1}(z)$$

where $\pi_{n-1}(z)$ is the $(n-1)^{th}$ monic OP wrt $w(z)$ and

$p_{n-1}(z) = \gamma_{n-1}\pi_{n-1}(z)$ is the normalized OP. Then

$$\frac{-1}{2\pi i}\int_{\mathbb{R}} h(s)\,w(s)\,s^{n-1}\,ds$$

$$= \gamma_{n-1}^2\int_{\mathbb{R}} \pi_{n-1}(s)\,w(s)\,s^{n-1}\,ds$$

$$= \gamma_{n-1}^2\int \pi_{n-1}^2(s)\,w\,ds \quad , \text{ as } \int \pi_{n-1}(s)\,s^j\,ds = 0$$
for $0 \le j \le n-2$.

$$= \int p_{n-1}^2(s)\,w(s)\,ds$$

$$= 1.$$

Hence $\ell(z) = h(z) - Y_{21}(z)$ is a polynomial of degree $n-1$

with $\int \ell(s)\,s^j\,w(s)\,ds = 0$ for $0 \le j \le n-1$.

Thus $\int |\ell(s)|^2 w(s)\,ds = 0$ and so $\ell \equiv 0$

It follows that

$$Y_{21}(z) = -2\pi i\, \gamma_{n-1}^2\, \pi_{n-1}(z)$$

This completes the proof that if the RHP has a solution $Y(z)$, then necessarily it is given by (189.3). Conversely, if we define $Y(z)$ by (189.3) (the OP's $\pi_k(z)$, $k \geq 0$, exist by a general argument), then it is a straightforward exercise (use (193.2)–(193.41)) to verify that, for any $n \geq 0$, $Y(z) = Y^{(n)}(z)$ solves the RHP (191.1) in the strong sense. This completes the proof of the FIK Theorem 189.2.

The RHP representation of the OP's in Th^m 189.2 should be viewed as a non-commutative representation of the OP's which generalizes the integral rep's which ∃ for certain special OP's, such as the Hermite polynomials (see (181.31)): the <u>very significant</u> difference is that an RHP rep. ∃ for <u>all</u> OP's.

What is not clear is why the RHP rep. helps to evaluate the asymptotics of the OP's. Such an evaluation is possible <u>because there is a non-commutative version</u> of the classical steepest-descent for integrals, which applies to RHP's.

We will now show how the method works for the above RHP for OP's.

But first we note that the FIK RHP also provides formulae for the basic quantities associated with OP's viz the normalization coefficients γ_n for the π_n's, and the <u>recurrence coefficients</u> a_n, b_n for the OP's. Note that OP's $\{p_n\}$ <u>always</u> solve a 3-term recurrence relation

(199.1) $$b_n\, p_{n+1}(z) + (a_n - z)\, p_n(z) + b_{n-1}\, p_{n-1}(z) = 0, \quad n \geq 0$$

where $a_n \in \mathbb{R}$, $b_n > 0$, for $n \geq 0$ and $b_{-1} \equiv 0$.

Indeed $z\, p_n(z)$ is a polynomial of degree $n+1$ and so can be expressed in the form

$$z\, p_n(z) = \sum_{j=0}^{n+1} \sigma_j\, p_j(z)$$

for suitable constants σ_j. For $k < n-1$

$$0 = \int \underbrace{z\, p_k(z)}_{\deg \leq n-1} p_n(z)\, w(z)\, dz$$

$$= \int p_k(z)\; z\, p_n(z)\, w\, dz = \sum_{j=0}^{n+1} \sigma_j \int p_k\, p_j\, w\, dz = \sigma_k$$

and so $$z\, p_n(z) = \sigma_{n+1}\, p_{n+1}(z) + \sigma_n\, p_n(z) + \sigma_{n-1}\, p_{n-1}(z)$$

It is a simple <u>exercise</u> to see that this relation takes the form (199.1) with $b_n > 0$ as advertised.

From (189.3)

$$Y_{12}^{(n)}(z) = \int_{\mathbb{R}} \frac{\pi_n(s)\, w(s)}{s-z} \frac{ds}{2\pi i}$$

$$= -\frac{1}{2\pi i z} \int \pi_n(s)\, w(s) \left(1 + \frac{s}{z} + \frac{s^{n-1}}{z^{n-1}} + \frac{s^n}{z^n} + \dots\right)$$

$$= -\frac{1}{2\pi i z^{n+1}} \int \pi_n(s)\, w(s) \left(s^n + O\left(\tfrac{1}{z}\right)\right)$$

$$= \left(Y_1^{(n)}\right)_{12} z^{-n-1} + \dots$$

where

$$\left(Y_1^{(n)}\right)_{12} = -\frac{1}{2\pi i} \int \pi_n(s)\, s^n\, w(s)\, ds$$

$$= -\frac{1}{2\pi i} \int \pi_n^2(s)\, w(s)\, ds$$

$$= -\frac{1}{2\pi i \gamma_n^2}$$

Hence

(200.1) $$\gamma_n^{-2} = -2\pi i \left(Y_1^{(n)}\right)_{12}$$

and similarly (exercise)

(200.2) $$a_n = \left(Y_1^{(n)}\right)_{11} - \left(Y_1^{(n+1)}\right)_{11}, \quad b_{n-1}^2 = \left(Y_1^{(n)}\right)_{12} \left(Y_1^{(n)}\right)_{21}$$

Thus γ_n, a_n and b_n can be read off directly from the solution of the RHP

$$Y^{(n)}(z)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n}\end{pmatrix} = I + z^{-1} Y_1^{(n)} + O(z^{-2})$$

In particular, once the asymptotic behavior of $Y^{(n)}(z)$ is known, we can simply read off the asymp. behavior of γ_n, a_n, b_n from (200.1) (200.2).

The following function (known for historical reasons as the "g-function") plays a critical role in the analysis of the RHP for OP's:

(201.1) $$g(z) = \int \log(z-s)\, d\mu^{eq}(s)$$

where $d\mu^{eq}$ is the eqm. meas. for the OP's wrt $w(s)ds$ $\left(= e^{-NV(s)}ds\right)$. For each s, $\log(z-s)$ is the canonical branch of the log, analytic in $\mathbb{C}\setminus(-\infty, s]$

and positive for $z > s$,

(202.1) $$\log(z-s) = \log|z-s| + i\arg(z-s) \quad , \quad z \in \mathbb{C}\setminus(-\infty, s]$$

where $-\pi < \arg(z-s) < \pi$.

Suppose that $d\mu^{ed}$ is supported on a single interval $I = (a,b)$ and $d\mu^{eq}(s) = \psi(s)\,ds$ for some cont function $\psi(s)$; $\psi(s) > 0$ on I. Then $g(z)$ has, in particular, the following properties: recall

$$2\int \log|x-y|\, d\mu^{ed}(y) - V(x) - \ell \le 0 \quad \text{with equality on } I$$

(202.2) $$g_+(z) + g_-(z) - V(z) - \ell = 0 \qquad \text{for } z \in I$$

(202.3) $$g_+(z) - g_-(z) \in i\mathbb{R}$$

$$\text{and} \quad i\frac{d}{dz}\left(g_+(z) - g_-(z)\right) > 0 \qquad \text{for } z \in I$$

(202.4) $$g_+(z) + g_-(z) - V(z) - \ell < 0 \qquad \text{for } z \in \mathbb{R}\setminus\bar{I}$$

(202.5) $$e^{g_+(z) - g_-(z)} = 1 \qquad \text{for } z \in \mathbb{R}\setminus I$$

Indeed, for $z \in I$

$a \quad z \quad b$

$$g_+(z) + g_-(z) - V(z) - \ell = 2\int \log|z-s|\, d\mu^{ed}(s) - V(z) - \ell = 0$$

which proves (202.2). Also for $z \in I$

$$g_+(z) - g_-(z) = \int_z^b [i\pi - (-i\pi)]\, d\mu^{eq}(s)$$

$$= 2i\pi \int_z^b d\mu^{eq}(s) \quad \in i\mathbb{R}_+$$

and $\quad i \frac{d}{dz}\left(g_+(z) - g_-(z)\right) = 2\pi \psi(z) \quad > 0$,

which proves (202.3)

For $z \in \mathbb{R} \setminus I$

$$g_+(z) + g_-(z) - V(z) - \ell = 2\int \log|z-s|\, d\mu^{eq}(s) - V(z) - \ell$$
$$< 0$$

which proves (202.4) and for $z > b$.

$g_+(z) = g_-(z)$, while for $z < a$ $\quad g_+(z) - g_-(z) = 2\pi i$.

This proves (202.5).

<u>Exercise:</u> Show that, conversely, properties (202.2) – (202.5) for $g(z) = \int \log(z-s)\, d\mu(s)$, $\operatorname{supp} d\mu = (a,b)$, $d\mu(s) = w(s)ds$, imply that $\mu = \mu^{eq}$ for $e^{-NV(x)}dx$.

Properties (202.2) – (202.5) play a crucial role in the steepest-descent method for RHP's.