Lecture 13

In particular, we know from our previous calculations that for $V(x) = t x^{2m}$, $t > 0$, $m = 1, 2, \ldots$,

$d\mu^{eq}$ is supported on a single interval

$$I = (-a, a)$$

where $a = \left(m t \prod_{l=1}^{m} \frac{2l-1}{2l}\right)^{-\frac{1}{2m}}$ (see (173.1))

and

$$d\mu^{eq}(x) = \frac{mt}{\pi} \sqrt{a^2 - x^2}\, h_1(x)\, \chi_{(-a,a)}(x)\, dx$$
$$= \psi(x)\, dx$$

where $h_1(x)$ is given by (172.2)

In particular, relations (202.2) – (202.5) hold

for $g(z) = \int_I \log(z - s)\, \psi(x)\, dx$ with

$$l = 2\int_{-a}^{a} \log(a - s)\, \psi(x)\, dx - V(a).$$

For an explicit evaluation of l in the case $m = 1$, $V = x^2$, see ref [2], and in the general case $m = 1, 2, \ldots$, see Deift – Kriecherbauer – McLaughlin – Venakides – Zhou, CPAM

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Recall the RHP for $\pi_n(z)$: $\Sigma = \mathbb{R}$, $v = \begin{pmatrix} 1 & e^{-nV} \\ 0 & 1 \end{pmatrix}$

$\xrightarrow{+} \mathbb{R}$

(205.1)
- $Y(z)$ 2×2 anal. in $\mathbb{C}\setminus\mathbb{R}$, cont. up to bdr
- $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & e^{-nV} \\ 0 & 1 \end{pmatrix}$, $z \in \mathbb{R}$.
- $Y(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \to I$ as $z \to \infty$ in $\mathbb{C}\setminus\mathbb{R}$.

Bearing in mind the classical steepest descent method for scalar integrals, the question is this: which point in $\mathbb{C}$ plays the role of the critical point where the leading order contribution to the integral is located? Looking at (205.1), this is not at all clear. The *first step* in the non-commutative steepest-descent method for RHP's of this type is to *precondition* (205.1) in such a way so as to transform the RHP into a standard RHP that is normalized at ∞. This is done in the following way. Let $\tilde{g}(z)$ be any function that is

(206.1)
- anal. in $\mathbb{C}\setminus\mathbb{R}$ & cont. up to the bdry
- $\tilde{g}(z) \sim \log z + o(1)$ as $z\to\infty$ in $\mathbb{C}\setminus\mathbb{R}$.

Set

$$U(z) = Y(z)\begin{pmatrix} e^{-n\tilde{g}(z)} & 0 \\ 0 & e^{+n\tilde{g}(z)} \end{pmatrix} = Y(z)\, e^{-n\sigma_3 \tilde{g}(z)}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the 3rd Pauli matrix.

Then

(206.2)
- $U(z)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$ & cont. up to the bdry
- $U_+(z) = Y_+(z)\, e^{-n\sigma_3\tilde{g}_+(z)} = Y_-(z)\begin{pmatrix} 1 & e^{-nV(z)} \\ 0 & 1 \end{pmatrix} e^{-n\sigma_3\tilde{g}_+(z)}$

$$= U_-(z)\,\tilde{v}(z)$$

where $\tilde{v}(z) = e^{n\sigma_3\tilde{g}_-}\begin{pmatrix} 1 & e^{-nV} \\ 0 & 1 \end{pmatrix} e^{-n\sigma_3\tilde{g}_+}$

$$= \begin{pmatrix} e^{n(\tilde{g}_- - \tilde{g}_+)} & e^{n(\tilde{g}_+ + \tilde{g}_- - V)} \\ 0 & e^{n(\tilde{g}_+ - \tilde{g}_-)} \end{pmatrix}, \quad z\in\mathbb{R}.$$

(206.2) $\bullet \quad U(z) = Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^{n} \end{pmatrix} \begin{pmatrix} e^{n(\log z - \tilde{g}(z))} & 0 \\ 0 & e^{-n(\log z - \tilde{g}(z))} \end{pmatrix}$

$\to I$ as $z \to \infty$

Thus $U(z)$ solves the normalized RHP $(\Sigma = \mathbb{R}, \tilde{v}(z))$.

Now: conjugate $U(z)$ further; for any constant $\tilde{\ell}$, set

(207.1) $$W(z) = e^{-\frac{n}{2}\tilde{\ell}\sigma_3}\, U(z)\, e^{\frac{n}{2}\tilde{\ell}\sigma_3}.$$

Then $W(z)$ solves the normalized RHP (Σ, v_W)

where

(207.2) $$v_W = \begin{pmatrix} e^{n(\tilde{g}_- - \tilde{g}_+)} & e^{n(\tilde{g}_+ + \tilde{g}_- - V - \tilde{\ell})} \\ 0 & e^{n(\tilde{g}_+ - \tilde{g}_-)} \end{pmatrix}$$

<u>Step 2</u> Now choose

$$\tilde{g}(z) = g(z) = \int \log(z-s)\, \psi(s)\, ds$$

$= \log z \int d\mu^{eq} + O(1/z) = \log z + o(1)$ (as $z \to \infty$)

and

$$\tilde{\ell} = \ell = 2\int_{-a}^{a} \log(a-s)\, \psi(x)\, dx - V(a).$$

Observe that with these choices, the relations

(202.2) (202.4) and (202.5) imply that v_W takes

The form

$$(208.1)\qquad \begin{pmatrix} 1 & e^{nT} \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} e^{-nS} & 1 \\ 0 & e^{nS} \end{pmatrix} \quad \begin{pmatrix} 1 & e^{+nT} \\ 0 & 1 \end{pmatrix}$$

$$\overline{\qquad\qquad\quad -a \qquad\qquad\qquad\qquad\qquad\qquad a \qquad\qquad\quad}$$

(208.2) where $T = g_+ + g_- - V - \ell < 0$, $|z| > a$

and $S = g_+ - g_-$, $-a < z < a$.

At this point, as $T(z) < 0$ for $z > a$ and $z < -a$, it is clear that as $n \to \infty$ the RHP for W <u>localizes</u> to the interval $(-a, a) = I$.

On the interval I,

$$S(z) = g_+ - g_- = 2\pi i \int_z^a \psi(s)\,ds \in i\mathbb{R}$$

and so e^{-nS} is oscillatory as $n \to \infty$. What we now try to do is to <u>deform</u> e^{-nS} into the complex plane so that it becomes exponentially decreasing, and at the same time deform e^{nS} into a <u>different</u> part of the plane, so that it is also decreasing. To do this we must clearly separate e^{-ns} and e^{ns} algebraically. This is done in the following way: observe that for $z \in I$,

$$S = 2\pi i \int_z^a \frac{mt}{\pi} \sqrt{a^2 - s^2}\, h_1(s)\, ds$$

$$= 2mt \int_z^a (s^2 - a^2)_+^{\frac{1}{2}}\, h_1(s)\, ds.$$

so $S(z)$ has an analytic continuation to $\mathbb{C}_+$

$$S(z) = 2mt \int_z^a (s^2 - a^2)^{\frac{1}{2}} h_1(s)\, ds ,$$

where $z \to a$ is any path in $\mathbb{C}_+$. Also S has an anal. cont.

$$S(z) = -2mt \int_z^a (s^2 - a^2)^{\frac{1}{2}} h_1(s)\, ds ,$$

where $z \to a$ is any path in $\mathbb{C}_-$.

Write

$$S = \alpha + i\beta \qquad \text{where } \alpha, \beta \text{ are the}$$

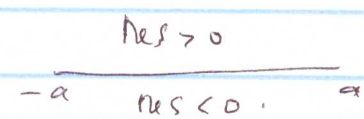
real & imag. parts of S respectively. For $z \in I$, we have $\alpha(z)=0$, and from (202.4) we see that $i\frac{\partial}{\partial x} i\beta = i\frac{\partial}{\partial x} S = i\frac{\partial}{\partial x}(g_+ - g_-) > 0$

Thus $\frac{\partial \beta}{\partial x} < 0$, and hence by the Cauchy Riemann condition, $\frac{\partial \alpha}{\partial y} = -\frac{\partial \beta}{\partial x} > 0$. It follows that

(210.1) $$\operatorname{Re} S(x+iy) > 0 \quad \text{for } -a<x<a,\ y>0 \text{ small}$$

and

(210.2) $$\operatorname{Re} S(x+iy) < 0 \quad \text{for } -a<x<a,\ y<0 \text{ small.}$$



Thus we want to deform e^{-nS} into $\mathbb{C}_+$ and e^{nS} into $\mathbb{C}_-$.

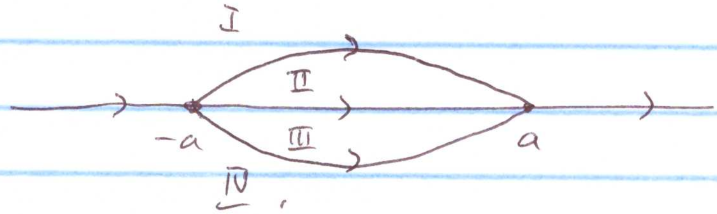
Now note that we can separate e^{-nS} & e^{nS} by using the following factorization.

(210.3) $$\begin{pmatrix} e^{-nS} & 1 \\ 0 & e^{nS} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{nS} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-nS} & 1 \end{pmatrix}.$$

Note that is just the upper/lower factorization of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} / \begin{pmatrix} e^{-nS} & 1 \\ 0 & e^{nS} \end{pmatrix}$

$= \begin{pmatrix} 1 & e^{ns} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-ns} & 1 \end{pmatrix}$. Such factorizations play a basic role in the st. descent method

Step 3 Now open up the "lenses", $\Sigma \to \hat{\Sigma}$



$\hat{\Sigma}$

to obtain a new oriented contour $\hat{\Sigma}$. Note that $\mathbb{C} \setminus \hat{\Sigma}$ has 4 components I, II, III, IV.

Define $\hat{W}(z)$ as follows:

(211.1) For $z \in \mathrm{I} \cup \mathrm{IV}$,

$$\hat{W}(z) \equiv W(z).$$

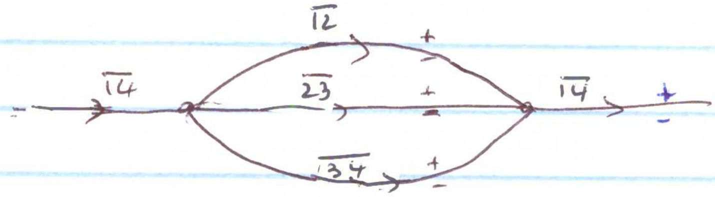
(211.2) For $z \in \mathrm{II}$,

$$\hat{W}(z) \equiv W(z) \begin{pmatrix} 1 & 0 \\ e^{-nS(z)} & 1 \end{pmatrix}^{-1}.$$

(211.3) For $z \in \mathrm{III}$,

$$\hat{W}(z) \equiv W(z) \begin{pmatrix} 1 & 0 \\ e^{nS(z)} & 1 \end{pmatrix}$$

Clearly $\hat{W}(z)$ is analy. in $\mathbb{C} \setminus \hat{\Sigma}$ and continuous up to the boundary. Denote the arcs in $\hat{\Sigma}$ as follows:



(212.1) For $z \in \overline{14}$, we clearly have $\hat{W}_+(z) = \hat{W}_-(z)\, v_W(z)$

(212.2) For $z \in \overline{12}$: $\hat{W}_+(z) = W_+(z) = W_-(z)$

$$= \hat{W}_-(z) \begin{pmatrix} 1 & 0 \\ e^{-ns} & 1 \end{pmatrix}.$$

(212.3) For $z \in \overline{23}$: $\hat{W}_+(z) = W_+(z) \begin{pmatrix} 1 & 0 \\ e^{-ns} & 1 \end{pmatrix}^{-1}$

$$= W_-(z) \begin{pmatrix} 1 & 0 \\ e^{ns} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \hat{W}_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

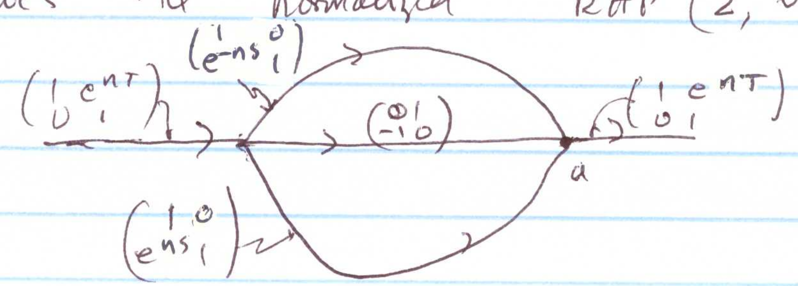
(212.4) For $z \in \overline{34}$: $\hat{W}_+(z) = W_+(z) \begin{pmatrix} 1 & 0 \\ e^{ns} & 1 \end{pmatrix}$

$$= W_-(z) \begin{pmatrix} 1 & 0 \\ e^{ns} & 1 \end{pmatrix}$$

$$= \hat{W}_- \begin{pmatrix} 1 & 0 \\ e^{ns} & 1 \end{pmatrix}$$

Thus $\hat{W}(z)$ solves the normalized RHP $(\hat{\Sigma}, \hat{v})$

where $\hat{v}$ has form:



As $\operatorname{Re} s \gtrless 0$ in $\mathbb{C}_+/\mathbb{C}_-$ resp, and $T<0$ for $z \in \mathbb{R}\setminus I$, we see that the RHP $(\hat{\Sigma}, \hat{v})$ localizes to the constant coefficient normalized RHP

$$\left(\Sigma_\infty = (-a, a)\ ,\ v_\infty = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$

It is crucial to note that all the RHP's in Steps I, II and III are equivalent, in the sense that if we can solve any one of them, then we can obtain the solution of any of the others just by an algebraic transformation.

The solution of the RHP $\left(\Sigma_\infty, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$ can be obtain explicitly. Indeed observe that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1}$$

Hence if we seek $W_\infty(z)$ analy. in $\mathbb{C}\setminus(-a,a)$ st $(W_\infty)_+ = (W_\infty)_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $(W_\infty)_+ \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = (W_\infty)_- \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$\Rightarrow$ Thus $W_\infty(z)\begin{pmatrix}1&1\\i&-i\end{pmatrix}$, and hence $\begin{pmatrix}1&1\\i&-i\end{pmatrix}^{-1}W_\infty(z)\begin{pmatrix}1&1\\i&-i\end{pmatrix}$, solves a pair of scalar RHP's and hence can be solved explicitly by Plemelj's formula (cf p123: note that a scalar RHP $\phi_+ = \phi_- v$ becomes an additive RHP, if we take logarithms, which we can do only in the scalar case,

$(\log\phi)_+ = (\log\phi)_- + \log v$). Hence

$$\begin{pmatrix}1&1\\i&-i\end{pmatrix}^{-1} W_\infty(z) \begin{pmatrix}1&1\\i&-i\end{pmatrix} = \begin{pmatrix} e^{\frac{1}{2\pi i}\int_{-a}^{a}\frac{\log i\, ds}{s-z}} & 0 \\ 0 & e^{\frac{1}{2\pi i}\int_{-a}^{a}\frac{\log(-i)\, ds}{s-z}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\frac{1}{4}\log\frac{z-a}{z+a}} & 0 \\ 0 & e^{-\frac{1}{4}\log\frac{z-a}{z+a}} \end{pmatrix}$$

$$= \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$$

where

(214.1) $$\beta(z) = \left(\frac{z-a}{z+a}\right)^{\frac{1}{4}}, \qquad z \in \mathbb{C}\setminus[-a,a]$$

$\beta(z)$ is anal. in $\mathbb{C}\setminus[-a,a]$ and $\beta(z) > 0$ for $z > a$.

Thus

$$W_\infty(z) = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} \frac{1}{-2i}$$

$$= \begin{pmatrix} \beta & \beta^{-1} \\ i\beta & -i\beta^{-1} \end{pmatrix} \begin{pmatrix} +i & +1 \\ +i & -1 \end{pmatrix} \frac{1}{+2i}$$

ie

(215.1) $$W_\infty(z) = \begin{pmatrix} \frac{\beta+\beta^{-1}}{2} & \frac{\beta-\beta^{-1}}{2i} \\ \frac{\beta^{-1}-\beta}{2i} & \frac{\beta+\beta^{-1}}{2} \end{pmatrix}$$

Note that $W_\infty(z)$ does not solve the RHP $(\Sigma_\infty, v_\infty)$ in the classical sense: it has a 4th root singularity at $z = \pm a$. However $W_\infty(z)$ is the unique solution of the RHP $(\Sigma_\infty, v_\infty)$ in an appropriate L^2 sense. Indeed if $\hat{W}_\infty$ is a second solution with an at-worst L^2 singularity at $z = \pm a$, then, as before, $\hat{W}_\infty W_\infty^{-1}(z)$ has no jumps across $(-a,a) \cup (a,\infty) \cup (-\infty,-a)$

(Note that (from (215.1)) $\det W_\infty = 1$: this of course also follows from the fact that $\det v_\infty = 1$, as before (but of course can be seen directly from (215.1)).

Hence $\hat{W}_\infty W_\infty^{-1}$ has at worst isolated singularities at $\pm a$.

But as $\hat{W}_\infty(z)$ and $W_\infty^{-1}(z)$ have L^2 sing's at $\pm a$, $\oint (s \pm a)^k \hat{W}_\infty W_\infty^{-1}(s)\, ds = 0$, $k \geq 0$, for any small circle about $\pm a$.

insert from 216+ 216++

Hence $\hat{W}_\infty W_\infty^{-1}(z)$ has a removable singularity at $\pm a$, which implies that $\hat{W}_\infty W_\infty^{-1}(z)$ is in fact entire, & so as $\hat{W}_\infty(z), W_\infty(z) \to I$, by Liouville we must have $\hat{W}_\infty = W_\infty$. Uniqueness!

Now we anticipate that $\hat{W} \to W_\infty$ as $n \to \infty$. The situation is a familiar one: suppose we are trying to solve an elliptic problem in a region Ω in the plane:

$$
(216.1) \quad \left\{ \begin{array}{ll} \nabla \cdot a(\varepsilon) \nabla f = 0 & \text{in } \Omega \\ f = F & \text{on } \partial\Omega \end{array} \right.
$$

Insert on p216

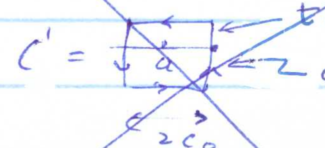
The argument is as follows:

Suppose $Z(z) = \hat{W}_\infty(z) W_\infty^{-1}(z)$ is in L^1 near $z = a$ in the sense that $\int_\Sigma |Z(s)|\,|ds| \leq C < \infty$ for small contours Σ near a. Then for any $k \geq 0$, any for $\Sigma = \{|z-a| = \varepsilon\}$, we have

$$(216+.1) \qquad \left| \int_{|s-a|=\varepsilon} Z(s)(s-a)^k\,ds \right| \leq \varepsilon^k \int_{|s-a|=\varepsilon} |Z(s)|\,ds \leq C\varepsilon^k$$

But as $Z(s)(s-a)^k$ is analytic in the punctured disk $|z-a| > 0$, ~~Thus~~ the LHS in (216+.1) is indep of ε. Letting $\varepsilon \downarrow 0$, we conclude the $\int_{|s-a|=\varepsilon_0} Z(s)(s-a)^k = 0$ for any fixed, small $\varepsilon_0 > 0$, and $k \geq 1$. Hence $Z(z)$ is of the form $c'/(z-a)$ + analytic for some const. c'. Now

~~we can deform the integral~~ ~~$\int_{|s-a|=\varepsilon} Z(s)\,ds = \int_{C'} Z(s)\,ds$.~~



~~For some small ε_0 and $t > 0$. Now keep ε_0 fixed and let $t \downarrow 0$.~~

for $\Sigma = \{a + it + x : c_0, t>0, -c_0<x<c_0\}$

$$-c_0+a+it \longrightarrow c_0+it+a$$

$$\dot{a}$$

we have

$$\int_\Sigma |Z(s)|\,ds$$

$$= |c'| \int_{-c_0}^{c_0} \left|\frac{1}{x+it}\right| dx + \text{bded}.$$

$$= |c'| \int_{-c_0/t}^{c_0/t} \frac{dy}{\sqrt{1+y^2}} + \text{bded}$$

Keeping c_0 fixed and letting $t \downarrow 0$ we conclude the

$$\int_\Sigma |Z(s)|\,|ds| \sim |c'| \log t^{-1} \to \infty$$

which contradicts $\int_\Sigma |Z(s)|\,ds \le c < \infty$, unless $c' = 0$

Thus the singularity of $\tilde{W}_\infty W_\infty^{-1}(z)$ at $z = \pm a$ is

removable.

Now suppose $a(\varepsilon) \to a(0)$ in some sense (For example $a(\varepsilon)$ may have oscillations, and the convergence is in the weak sense.) as $\varepsilon \downarrow 0$. Question: does the solution $f = f_\varepsilon$ of (216.1), converge in some sense to the solution $f = f_0$ of

$$\nabla \cdot a(0) \nabla f_0 = 0 \quad \text{in } \Omega$$
$$f_0 = F \quad \text{on } \partial\Omega$$

as $\varepsilon \downarrow 0$?

The answer is sometimes "no": eg. suppose we try to solve the simple equation.

$$H_\varepsilon f = \left(1 + \tfrac{1}{2}\sin(x/\varepsilon)\right) f(x) = 1 \quad , x \in \mathbb{R}.$$

Clearly H_ε converges weakly to 1 as $\varepsilon \downarrow 0$.

But

$$f = f_\varepsilon(x) = \frac{1}{1 + \frac{1}{2}\sin\frac{x}{\varepsilon}} \quad \cdots$$

$$= 1 - \tfrac{1}{2}\sin\tfrac{x}{\varepsilon} + \left(\tfrac{1}{2}\sin^2\tfrac{x}{\varepsilon}\right) + \cdots$$

$$= 1 - \tfrac{1}{2}\sin\tfrac{x}{\varepsilon} + \tfrac{1}{4}\left(1 - \cos\tfrac{2x}{\varepsilon}\right) + \cdots$$

$$\xrightarrow[\varepsilon \downarrow 0]{} 1 + \tfrac{1}{4} + \cdots \neq 1 = \frac{1}{H_0} 1 = f_0$$

More abstractly, if we are solving an equation $A_\varepsilon f = g$, and $A_\varepsilon \to A_0$ in norm, A_0 invertible, then $f = f_\varepsilon = \frac{1}{A_\varepsilon} g \to \frac{1}{A_0} g = f_0$.

But if $A_\varepsilon \to A_0$ weakly, or even strongly, then f_ε may not converge to f_0. This is precisely the situation we are facing with our RHP $(\hat{\Sigma}, \hat{v})$. Although $\hat{v}(z) \to v_\infty(z)$, $z \in \hat{\Sigma} \setminus \{\pm a\}$, the convergence is not uniform: it clearly becomes slower and slower as z approaches $\pm a$. Thus $\|\hat{v} - v_\infty\|_{L^\infty(\hat{\Sigma})} \not\to 0$ as $n \to \infty$, and as we will see shortly, it is precisely the L^∞ norm of $\hat{v} - v_\infty$ that controls the convergence of the RHP's. (See p220+1,…,+7, below ~~for~~ more details.)

Nevertheless, it turns out that indeed

$\hat{W}(z) \to W_\infty(z)$ as $n \to \infty$.