Retracing our steps we see that for z in region $\underline{\text{II}}$, say, of p211,

$$
\begin{aligned}
(219.1) \qquad Y(z) &= U(z)\, e^{n\sigma_3 g(z)} \\
&= e^{\frac{n}{2} l \sigma_3}\, W(z)\, e^{-\frac{n}{2} l \sigma_3}\, e^{n\sigma_3 g(z)} \\
&= e^{\frac{n}{2} l \sigma_3}\, \hat{W}(z) \begin{pmatrix} 1 & 0 \\ e^{-nS(z)} & 1 \end{pmatrix} e^{-\frac{n}{2} l \sigma_3}\, e^{n\sigma_3 g(z)} \\
&\underset{n\to\infty}{\sim} e^{\frac{n}{2} l \sigma_3} \begin{pmatrix} \frac{\beta+\beta^{-1}}{2} & \frac{\beta-\beta^{-1}}{2i} \\ \frac{\beta^{-1}-\beta}{2i} & \frac{\beta+\beta^{-1}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-nS} & 1 \end{pmatrix} e^{-\frac{n}{2} l \sigma_3}\, e^{n\sigma_3 g(z)}
\end{aligned}
$$

and we obtain, in particular, the asymptotics of $\pi_n(z) = Y_{11}(z)$ as $n \to \infty$. There are, of course, analogous formulae for z in the other 3 regions.

Insert (219+) →

A RHP (Σ, v) can be converted into a problem of singular integral equations on Σ in the following way.

Insert on p219

N.B.!!

We see here a fundamentally new phenomenon for the non-commutative steepest-descent method that is not present in the classical case: In the classical case, the leading order contribution to the integral comes from the stationary phase point, or a finite # of stationary phase points. But in the non-commutative case, we see that the leading order contribution to the RHP comes from a whole continuum of points, viz., $\Sigma = (-a, a)$.

Recall $v \in L^\infty$. Let

$$C_v f = \int_\Sigma \frac{f(s)\,(v(s) - I)}{s - z} \frac{ds}{2\pi i}$$

i.e. $C_v f = C_\Sigma[f(v - I)]$ where C_Σ is the Cauchy operator for Σ. Then for the contours we are considering (a finite union of smooth arcs, with a finite # of pts of self-intersection), and certainly for far more general contours (Carleson curves!), we have for $f \in L^2(\Sigma)$

$$C_{v\pm} f(z) = \lim_{z' \to z\pm} ((C_v f)(z'))$$

$$= \lim_{z' \to z\pm} C_\Sigma (f(v - I))(z) = C_{\Sigma\pm}(f(v - I))$$

$\exists$ ae as non-tangential limits.

Moreover $\| C_{v\pm} f \|_{L^2(\Sigma)} \le c \, \| f \|_{L^2}$

Also

$$C_{\Sigma+} f(z) - C_{\Sigma-} f(z) = f(z), \text{ a.e. } z \in \Sigma.$$

Insert on p222. Supplemental information on RHP's in the sense of $L^p(\Sigma)$, $1<p<\infty$: In order to make the asymptotic calculations for

220+1 → 220+7.

the RHP valid, we must define what it means to solve a RHP in the sense of $L^p(\Sigma)$, $1<p<\infty$.

(see, for example, P. Deift & X. Zhou CPAM 56, 2003, #8, 1029–1077, or ArXiv 0206224)

Let $1<p<\infty$.

We say a pair of (matrix-valued) functions $m_\pm$ in $L^p(\Sigma)$ belong to $\partial C_p(\Sigma)$ if $m_\pm = C^\pm_\Sigma h$ for some $h \in L^p(\Sigma)$.

Here $C^\pm_\Sigma$ are the boundary values of the Cauchy transform on Σ as before. Note that

$$h = C^+_\Sigma h - C^-_\Sigma h = m_+ - m_-$$

so h is uniquely determined by $m_\pm$.

We say that $m_\pm$ solves the normalized RHP (Σ, v) in the sense of if $m^\pm - I \in \partial C_p(\Sigma)$

(220+1.1) $$m_\pm = I + C^\pm h$$

for some $h \in L^p(\Sigma)$, and

(220+1.2) $$m_+ = m_- v \quad \text{a.e. on } \Sigma.$$

We say that

$$m(z) = I + C_\Sigma h(z)$$

$$= I + \frac{1}{2\pi i}\int_\Sigma \frac{h(s)}{s-z}\, ds$$

is the (analytic) <u>extension</u> of $m_\pm$ off Σ. Note that

- $m(z)$ is anal. in $\mathbb{C} \setminus \Sigma$
- $m_+ = m_- v$ a.e. on Σ
- $m(z) \to I$ (in some sense) as $z \to \infty$

so that $m_\pm$ above produces a solution of the RHP in the "natural" sense.

In order to solve the RHP (Σ, v) with $v, v^{-1} \in Gl(n, \mathbb{C})$ and $v - I \in L^p(\Sigma)$, we proceed as follows: Let $\mu \in I + L^p(\Sigma)$ solve the equation

(220+2.1) $$(1 - C_v)\mu = I = I_n$$

on Σ

Here $C_v h = C^-(h(v-I))$ and I is the func. identically $= I$ on Σ. Clearly $\|C_v h\|_p \le \|C^-\|_p \|v - I\|_\infty \|h\|_p$

We always assume Σ is Carleson so that $C^\pm$ are bded in $L^p(\Sigma)$, $\|C^\pm\|_p < \infty$. Hence $C_v \in \mathcal{L}(L^p)$.

More precisely, if we write $\mu = 1 + \lambda$, $\lambda \in L^p$

then (220+2.1) takes the form

(220+3.1) $$(1 - C_v)\lambda = C_v I = C^-(v - I) \in L^p$$

which is an equation in L^p. To solve (220+3.1), and hence (220+2.1), we need to know that

(220+3.2) $$(1 - C_v)^{-1} \quad \text{exists in } L^p(\Sigma).$$

Under this assumption, set

(220+3.3) $$m_\pm = I + C^\pm \mu(v - I)$$

Now, using (220+2.1)

(220+3.4(i)) $$\begin{aligned} m_+ &= I + C^+ \mu(v - I) \\ &= I + C^- \mu(v - I) + \mu(v - I) \\ &= I + C_v \mu + \mu(v - I) \\ &= \mu + \mu(v - I) = \mu v \end{aligned}$$

and

(220+3.4(ii)) $$m_- = I + C^-\mu(v - I) = I + C_v\mu = \mu$$

(220+3.5) and hence $m_+ = m_- v$. But as !

$$\mu(v-I) = (v-I) + \lambda(v-I) \in L^p$$

we see from (220+3.3) that $m_\pm \in I + \partial C_p$ and together with (220+3.4) we see that $m_\pm$ solve the normalized RHP (Σ, v) in the sense of L^p.

Thus the analysis of the RHP boils down to the analysis of the (singular integral) equation (220+2.1).

Now suppose that we have a sequence (Σ, v_n) of normalized RHPs on a fixed contour Σ, and suppose that as $n \to \infty$, $v_n \to v$, at least pointwise a.e. on Σ. How much more do we need to know about the convergence to ensure that the solutions $m_{n\pm}$ of the RHPs (Σ, v_n) converge to the solution $m_\pm$ of the (normalized) RHP (Σ, v)?

From 220+3.4 (i)(ii) we have

$$m_{n+} = \mu_n v_n \quad , \quad m_{n-} = \mu_n$$

$$m_{+} = \mu v \quad , \quad m_{-} = \mu$$

So to obtain convergence in $L^p(\Sigma)$ we would certainly need to know that

(220+5.1) $$\mu_n - \mu \to 0 \text{ in } L^p$$

As $m_{n+} - m_{+} = \mu_n v_n - \mu v = (\mu_n - \mu) v + (v_n - v)\mu$

We have $(v_n - v)\mu = (v_n - v) + (v_n - v)\lambda$, $\lambda \in L^p$

so $\|(v_n - v)\mu\|_{L^p} \to 0$ if

(a) $\cdot$ $v_n \to v$ in L^p, and by dominated convergen

(b) $\cdot$ $\sup_n \|v_n\| < \infty$.

Now from (220+3.1),

$$\mu_n - \mu = \lambda_n - \lambda = \frac{1}{1-C_{v_n}} C^-(v_n - I) - \frac{1}{1-C_v} C^-(v - I)$$

$$= \frac{1}{1-C_{v_n}} C^-(v_n - v) + \frac{1}{1-C_{v_n}} C^-(v - I) - \frac{1}{1-C_v} C^-(v-I)$$

$$= \frac{1}{1-C_{v_n}} C^-(v_n - v) + \left(\frac{1}{1-C_{v_n}} - \frac{1}{1-C_v}\right) C^-(v-I)$$

$$= \frac{1}{1-C_{v_n}} C^-(v_n - v) + \left(\frac{1}{1-C_{v_n}} C_{v_n - v} \frac{1}{1-C_v}\right) C^-(v-I)$$

Together with the above assumption (a), we have

$$\frac{1}{1-C_{\sigma_n}} C^{-}(\sigma_n - \sigma) \to 0 \quad \text{in } L^p(\Sigma)$$

if

(c) $\quad \left\| \frac{1}{1-C_{\sigma_n}} \right\| \le K < \infty \quad \forall\, n = 1, 2, \ldots$

In order for the second term to go to zero

$$\left(\frac{1}{1-C_{\sigma_n}} C_{\sigma_n - \sigma} \frac{1}{1-C_\sigma} \right) C^{-}(\sigma - I) \to 0 \quad \text{in } L^p(\Sigma)$$

we need, in addition to (c),

$$\| C_{\sigma_n - \sigma} \|_{L^p} \to 0$$

But $\quad \| C_{\sigma_n - \sigma} h \| = \| C^{-} h(\sigma_n - \sigma) \| \le \| C^{-} \|_p \|\sigma_n - \sigma\|_\infty \|h\|_p$

and so $\quad C_{\sigma_n - \sigma} \to 0 \quad$ if

(d) $\quad \|\sigma_n - \sigma\|_\infty \to 0$

Standard manipulations show that if

$$\|\sigma_n - \sigma\|_\infty \to 0$$

and $\quad (1 - C_\sigma)^{-1} \;\exists$, then $(1 - C_{\sigma_n})^{-1}$ for

n sufficiently large. Assembling the above calculations we

see that in order for $m_{n\pm} \to m_\pm$ in L^p we need

$$\cdot\ \|v_n - v\|_{L^p(\Sigma)} + \|v_n - v\|_{L^\infty} \to 0$$

(220+7.1) and

$$(1 - C_v)^{-1}\ 7 \text{ in } L^p(\Sigma).$$

This is why we cannot conclude that $\hat{W} \to W_\infty$ on p218: we do not know $\|\hat{v} - v_\infty\|_\infty \to 0$ as $n \to \infty$.

Assume as before that $v, v^{-1} \in L^\infty(\Sigma)$ and in addition,

(221.1) $$v - I \in L^2(\Sigma).$$

Suppose that

(221.1) $$1 - C_{v-} \quad \text{is invertible in } L^2(\Sigma)$$

and let

(221.2) $$\mu = I + \nu$$

solve the equation

(221.3) $$(1 - C_{v-})\mu = I$$

or more precisely,

$$(1 - C_{v-})\nu = C_{v-}I = C_{\Sigma,-}(v - I) \in L^2$$

Then

(221.4) $$m(z) \equiv I + \int_\Sigma \frac{\mu(s)\,(v(s) - I)}{s - z}\,\frac{ds}{2\pi i}, \quad z \in \mathbb{C} \setminus \Sigma$$

solves the normalized RHP (Σ, v). (Clearly $m(z)$ is anal. in $\mathbb{C} \setminus \Sigma$.) Indeed, for $z \in \Sigma$,

$$m_+(z) = I + C_{v+}(\mu)$$

$$= I + \mu(v-I) + C_{v-}(\mu) = \mu v$$

Also $m_-(z) = I + C_{v-}\mu = \mu$

Hence

$$m_+ = m_- v(z) \quad . \quad z \in \Sigma$$

Also, clearly, in some appropriate sense,

$$m(z) = I + O\left(\frac{1}{z}\right) \quad \text{as } z \to \infty,$$

by (221.4). Thus $m(z)$ solves the normalized RHP (Σ, v)

For $z \in \Sigma$, we see that

$$m_-(z) = \mu(z) = \frac{1}{1 - C_{v_-}} I$$

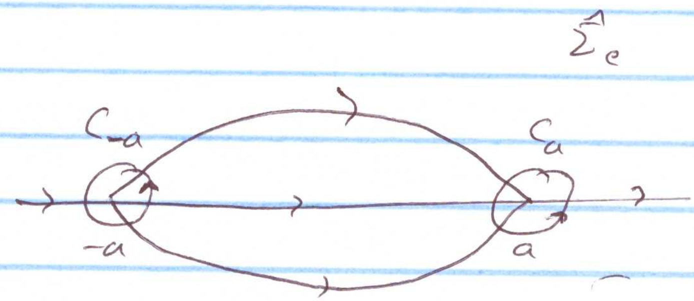
$$= I + \frac{1}{1 - C_{v_-}}\left(C_{\Sigma -}(v - I)\right)$$

So we see now clearly that we would need $\hat{v} \to v_\infty$ in $L^2 \cap L^\infty$-norm to conclude that $\hat{W} \to W_\infty$! We know $\hat{v} \to v_\infty$ in L^2, and point-wise a.e. on Σ, but not in $L^\infty(\Sigma)$!

For more details, see 220+1, ..., +7

In order to show $\hat{W} \to W_\infty$ we need more information about what is happening at the points $z = \pm a$. We proceed as follows.

Extend the RHP $(\hat{\Sigma}, \hat{v})$ for $\hat{W}$ on p 212, to a RHP $(\hat{\Sigma}_e, \hat{v}_e)$ by adding in 2 small circles $C_{\pm a}$ around $\pm a$



and set

(223.0)

$$\hat{W}_e(z) = \hat{W}(z), \quad z \in \mathbb{C} \setminus \hat{\Sigma}_e$$

Thus

$$\hat{v}_e(z) = \hat{v}(z) \quad \text{for } z \in \hat{\Sigma} \subset \hat{\Sigma}_e$$

and

$$\hat{v}_e(z) = I \quad \text{for } z \in C_a \cup C_{-a}$$

We then construct explicitly a <u>parametrix</u> $\hat{W}_p(z)$ with the following properties:

(223.1) $\hat{W}_p(z) = W_\infty(z)$ for z outside the "dumbell" region

(224.1) $$\hat{W}_p(z)_+ = \hat{W}_p(z)_- \hat{v}(z) \quad , \quad z \in (\hat{\Sigma} \cap \mathring{C}_{+a}) \cup (\hat{\Sigma} \cap \mathring{C}_{-a})$$

where $\mathring{C}_{\pm a}$ denote the interior of $C_{\pm a}$ resp:

(224.1) $$\hat{W}_{p+}(z) = \hat{W}_{p-}(z)\, \hat{v}_p(z) \quad , \quad z \in C_{\pm a}.$$

where $$\hat{v}_p(z) = I + O_n(z) \quad , \quad \| O_n \|_{L^\infty(C_{\pm a})} = O(\tfrac{1}{n})$$

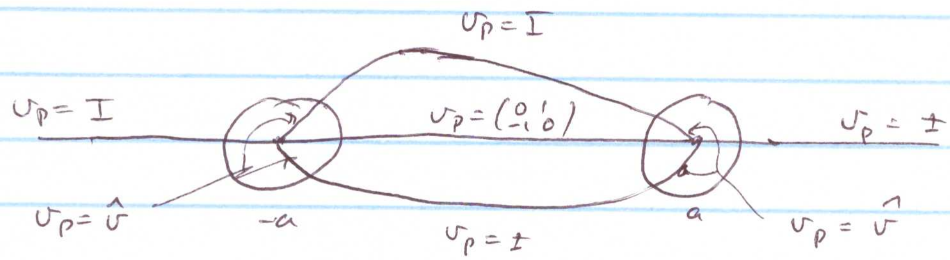
Note that on $C_{\pm a}$, $\hat{W}_{p-}(z) = W_\infty(z)$ and so (224.1)

is a condition on the construction of $\hat{W}_p(z)$ for $z \in (\mathring{C}_{+a} \cup \mathring{C}_{-a}) \setminus \hat{\Sigma}$

It follows, in particular from the above that

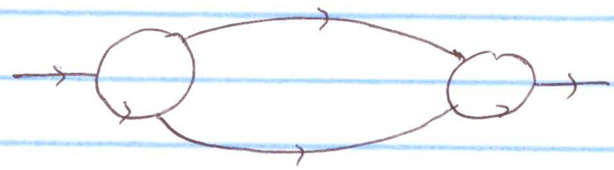
$\hat{W}_p(z)$ solves a normalized RHP $(\hat{\Sigma}_e, v_p)$

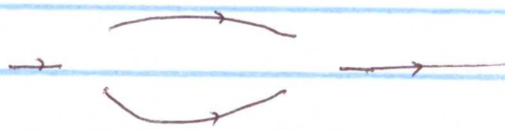
where



Thus $\hat{W}_e$ and $\hat{W}_p$ have precisely the same jumps

on $\longleftrightarrow$. Hence $R = \hat{W}_e \hat{W}_p^{-1}$ solves a normalized RHP with jumps on



On 

$$R_+ = \hat{W}_{e+} \hat{W}_{p+}^{-1} = \hat{W}_{e-} \hat{v}^{-1} \hat{W}_{p-}^{-1} = R_- \, \hat{W}_{p-} \hat{v}^{-1} \hat{W}_{p-}^{-1}$$

as $\hat{W}_p$ is anal. across these arcs. But as points on these arcs are at a finite distance from $\pm a$, $\|\hat{v}(z) - I\|_{L^\infty}$  is exponential decreasing, and hence the same is true for $v_R = \hat{W}_{p-} \hat{v}^{-1} \hat{W}_{p-}^{-1}$ $= W_\infty \hat{v}^{-1} W_\infty^{-1}$.

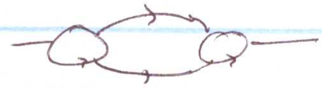
On the other hand, on 

$$R_+(z) = \hat{W}_{e+} \hat{W}_{p+}^{-1} = \hat{W}_{e-} \hat{W}_{p-}^{-1} \; \hat{W}_{p-} \hat{v}_p^{-1} \hat{W}_{p-}^{-1} = R_- \, v_R,$$

$(v_R = \hat{W}_{p-} \hat{v}_p^{-1} \hat{W}_{p-}^{-1} = W_\infty v_p^{-1} W_\infty^{-1})$

as $\hat{W}_e(z)$ is anal. across these arcs, and as $\|\hat{v}_p - I\|_{L^\infty(C_{\pm a})} = O(\frac{1}{n})$, the same is true for v_R.

It follows then that

$$\|v_R - I\|_{L^2 \cap L^\infty(}$$  $$) = O(\tfrac{1}{n})$$

and so

$$R = \hat{W}\hat{W}_p^{-1} \to I + O(\tfrac{1}{n}) \quad \text{in } L^2(\hat{\Sigma}_2)$$

ie

$$\hat{W} = (I + O(\tfrac{1}{n}))\hat{W}_p$$

As $\hat{W}_p$ is known explicitly from the construction, this provides the asymptotics of $\hat{W}$ as $n \to \infty$, and hence of $Y(z)$, by (219.1).

We will show how to construct $\hat{W}_p(z)$ (using Airy functions!) in $\mathbb{C}_{\pm a} \setminus \hat{\Sigma}$, and make the above argument precise, in the next lecture.