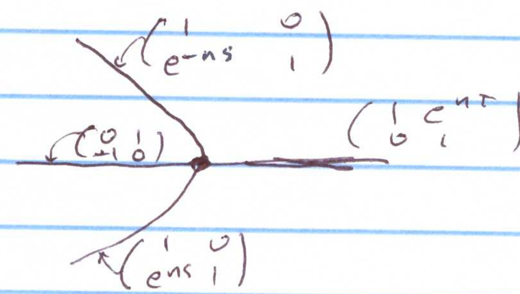
Lecture 15

In order to complete the above analysis of the asymptotic behavior of the OP's with weight $e^{-tNx^{2m}}dx$ we need, as described, to find an exact solution of the RHP $(\hat{\Sigma}_e, \hat{v})$ in a (small) open set O_a, $\partial O_a = C_{+a}$ around $z=a$, and similarly around $z=-a$

We only ~~only~~ consider $z=+a$: the case $z=-a$ is similar
In addition, on $C_{\pm a}$ matching as in (224.1),
We follow P.D.: OP's and Random Matrices: A R-Hilb approach

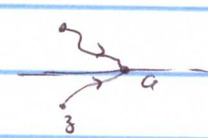
From p212, we have for $\hat{v}$



where (see p 209)

(227.1) $$s(z) = \pm 2mt \int_z^a (s^2-a^2)^{\frac{1}{2}} h_1(s)\, ds$$

in $\mathbb{C}_\pm$ resp.



and for $z > a$

$$T(z) = g_+ + g_- - V - \ell$$

$$= 2\int_{-a}^{a} \log(z-s)\,\psi(s)\,ds - V(z) - \ell$$

$$\therefore \quad \frac{\partial}{\partial z} T(z) = 2\int_{-a}^{a} \frac{1}{z-s}\,\psi(s)\,ds - V'(s)$$

$$= -2\pi i\, G(z) - V'(s)$$ (see p 160)

$$= 2mt\left(z^{2m-1} - (z^2-a^2)^{\frac{1}{2}} h_1(z)\right) - 2mt\, z^{2m-1})$$ (see p 172)

$$= -2mt\,(z^2-a^2)^{\frac{1}{2}}\, h_1(z)$$

(228.1) $$\therefore \quad T(z) = -2mt\int_a^z \underbrace{(s^2-a^2)^{\frac{1}{2}} h_1(s)}_{>0}\, ds$$

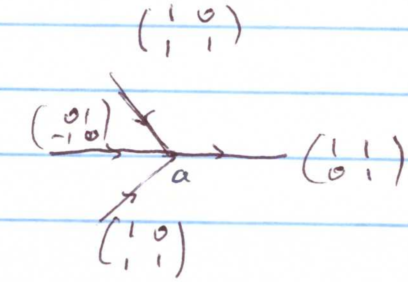
Set

(228.2) $$\phi(z) \equiv tm\int_a^z (s^2-a^2)^{\frac{1}{2}} h_1(s)\, ds, \qquad z \in \mathbb{C}\setminus(-\infty, a]$$

$$\phi(z) > 0 \quad \text{for } z > a$$

ϕ is analytic in its domain

Let $v^\#(z)$ be the jump matrix on $\Sigma_e \cap O_a$ given by



$v^\#$

and consider

$$v^\#(z) = e^{-n\phi_-\sigma_3}\ v^\#\ e^{n\phi_+\sigma_3}.$$

For $z > a$ $\longrightarrow$

$$v^\#(z) = e^{-ntm\left(\int_a^z (s^2-a^2)^{\frac{1}{2}} h_1\,ds\right)\sigma_3}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} e^{ntm\left(\int_a^z (s^2-a^2)^{\frac{1}{2}} h_1\,d\right.}$$

$$= \begin{pmatrix}1 & e^{nT}\\ 0 & 1\end{pmatrix}, \quad \text{by } (228.1)$$

For $z \in$

$$v^\#(z)\quad e^{-ntm\left(\int_a^z (s^2-a^2)^{\frac{1}{2}} h_1\,ds\right)\sigma_3}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} e^{ntm\left(\int_a^z (s^2-a^2)^{\frac{1}{2}} h_1\,ds\right)\sigma_3}$$

$$= \begin{pmatrix}1 & 0\\ e^{2ntm\left(\int_a^z (z^2-a^2)^{\frac{1}{2}} h_1\,ds\right)} & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ e^{-ns} & 1\end{pmatrix}, \quad \text{by } (227.1)$$

For $z \in$ $\qquad v^\#(z) = \begin{pmatrix}1 & 0\\ e^{2ntm\int_a^z (z^2-a^2)^{\frac{1}{2}} h_1\,ds} & 1\end{pmatrix}$

$$= \begin{pmatrix}1 & 0\\ e^{ns} & 1\end{pmatrix}, \quad \text{by } (227.1)$$

And for $z \in \longrightarrow_a$

$$v^{\#}(z) = e^{-ntm\left(\int_a^z (s^2-a^2)^{\frac{1}{2}} h_1 ds\right)_- \sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times e^{ntm\left(\int_a^z (s^2-a^2)^{1/2} h_1 ds\right)_+ \sigma_3}$$

$$= \begin{pmatrix} 0 & e^{-ntm\left(\left(\int_a^z (s^2-a^2)^{\frac{1}{2}} h_1 ds\right)_- + \left(\int_a^z (s^2-a^2)^{\frac{1}{2}} h_1 ds\right)_+\right)} \\ -e^{ntm\left(\left(\int_a^z (s^2-a^2)^{\frac{1}{2}} h_1 ds\right)_- + \left(\int_a^z (s^2-a^2)^{\frac{1}{2}} h_1 ds\right)_+\right)} & 0 \end{pmatrix}.$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

as $\left(\int_a^z (s^2-a^2)^{\frac{1}{2}} h_1 ds\right)_- + \left(\int_a^z (s^2-a^2)^{\frac{1}{2}} h_1 ds\right)_+$

$$= \int_a^z \left[(s^2-a^2)_+^{\frac{1}{2}} + (s^2-a^2)_-^{\frac{1}{2}} \right] h_1 ds$$

$= 0$ for $s \in (-a, a)$.

Thus

(230.0) $$v^{\#}(z) = \hat{v}(z) = e^{-n\phi_- \sigma_3} v^{\#} e^{n\phi_+ \sigma_3}$$

Thus in O_a

$$\hat{W}_+ = \hat{W}_- e^{-n\phi_- \sigma_3} v^{\#} e^{n\phi_+ \sigma_3}$$

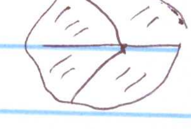
or

(230.1) $$L_+ = L_- v^{\#}$$

where

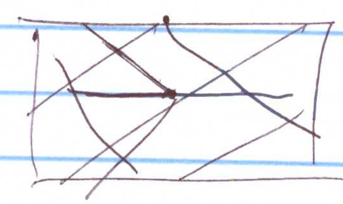
(230.2) $$L = \hat{W} e^{-n\phi \sigma_3}$$

We now try to solve the RHP in O_a with jump matrix $v^{\#}$:



$$(231.1)\quad \begin{cases} \cdot\ Z(z) \text{ analytic in } O_a \setminus \Sigma_e \\ \cdot\ Z_+ = Z_- v^{\#} \text{ on } \Sigma_e \cap O_a \equiv \Sigma_a \\ \cdot\ Z e^{n\phi\sigma_3} = W_\infty (I + O(\tfrac{1}{n})) \text{ on } C_a = \partial O_a \text{ as } n \to \infty. \end{cases}$$

~~Without loss we can assume that the arms of $\Sigma_a = \Sigma_e \cap O_a$ are segments of straight lines;~~



Recall (see Abramowitz & Stegun, pp 446 et seq) that $Ai(z)$ is the unique solution of Airy's equation

$$Ai''(z) = z\, Ai(z)$$

with asymptotics

$$(231.2)\quad Ai(z) = \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3}z^{3/2}} \left(1 + O\left(\frac{1}{z^{3/2}}\right)\right)$$

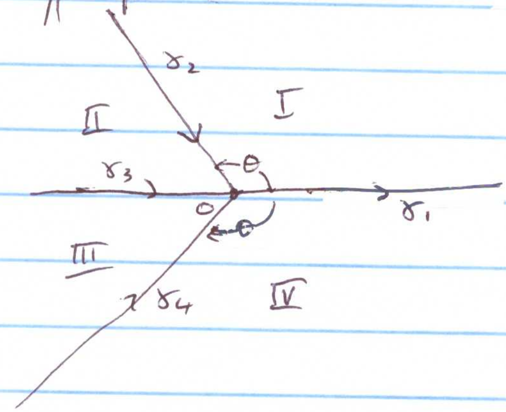
$$(231.3)\quad Ai'(z) = -\frac{1}{2\sqrt{\pi}} z^{1/4} e^{-\frac{2}{3}z^{3/2}} \left(1 + O\left(\frac{1}{z^{3/2}}\right)\right)$$

as $z \to \infty$, where $|\arg z| < \pi$. Although $Ai(z)$ is entire its asymptotics change as one crosses $(-\infty, 0)$.

The branches in (231.2) (231.3) are principal: i.e. if $z = re^{i\theta}$, $-\pi < \theta < \pi$, then $z^{1/4} = r^{1/4} e^{i\theta/4}$ and $z^{3/2} = r^{3/2} e^{i3\theta/2}$. The following relation (see Ab & Steg) is critical: for $\omega = e^{2\pi i/3}$, $\omega^3 = 1$,

(232.1) $$Ai(z) + \omega Ai(\omega z) + \omega^2 Ai(\omega^2 z) = 0 \qquad \forall z$$

Consider the oriented infinite contour Σ_∞ with straight lines passing through 0:



Here $0 < \theta < \frac{2\pi}{3}$ and is fixed.

with regions and arcs as displayed as above:

For $s \in$ I set

(232.2) $$\Psi(s) = \begin{pmatrix} Ai(s) & Ai(\omega^2 s) \\ Ai'(s) & \omega^2 Ai'(\omega^2 s) \end{pmatrix} e^{-i\frac{\pi}{6}\sigma_3}$$

For $s \in$ IV, set

(232.3) $$\Psi(s) = \begin{pmatrix} Ai(s) & -\omega^2 Ai(\omega s) \\ Ai'(s) & -Ai'(\omega s) \end{pmatrix} e^{-i\pi/6\,\sigma_3}.$$

For $s \in$ II, set

(232.4) $$\Psi(s) = \begin{pmatrix} Ai(s) & Ai(\omega^2 s) \\ Ai'(s) & \omega^2 Ai'(\omega^2 s) \end{pmatrix} e^{-i\pi/6\,\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

and for $z \in \mathrm{III}$

(233.1) $$\Psi(s) = \begin{pmatrix} Ai(s) & -\omega^2 Ai(\omega s) \\ Ai'(s) & -Ai'(\omega s) \end{pmatrix} e^{-i\frac{\pi}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Observe that for $s \in \gamma_1$

$$\Psi_-(s) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} Ai(s) & -\omega^2 Ai(\omega s) \\ Ai'(s) & -Ai'(\omega s) \end{pmatrix} e^{-i\frac{\pi}{6}\sigma_3} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} Ai(s) & -\omega^2 Ai(\omega s) \\ Ai'(s) & -Ai'(\omega s) \end{pmatrix} \begin{pmatrix} 1 & e^{-i\pi/3} \\ 0 & 1 \end{pmatrix} e^{-i\pi/6\,\sigma_3}$$

$$= \begin{pmatrix} Ai(s) & e^{-i\pi/3} Ai(s) - e^{\frac{i4\pi}{3}} Ai(\omega s) \\ Ai'(s) & e^{-i\pi/3} Ai'(s) - Ai'(\omega s) \end{pmatrix} e^{-i\pi/6\,\sigma_3}$$

$$= \begin{pmatrix} Ai(s) & e^{-i\pi/3}\left(Ai(s) + e^{2i\pi/3} Ai(\omega s) \right) \\ Ai'(s) & e^{-i\pi/3}\left(Ai'(s) + e^{i4\pi/3} Ai'(\omega s) \right) \end{pmatrix} e^{-i\frac{\pi}{6}\sigma_3}$$

$$\underset{(232.1)}{=} \begin{pmatrix} Ai(s) & e^{-i\pi/3}\left(-\omega^2 Ai(\omega^2 s) \right) \\ Ai'(s) & e^{-i\pi/3}\left(-\omega\, Ai'(\omega^2 s) \right) \end{pmatrix} e^{-i\frac{\pi}{6}\sigma_3}$$

$$= \begin{pmatrix} Ai(s) & Ai(\omega^2 s) \\ Ai'(s) & \omega^2 Ai'(\omega^2 s) \end{pmatrix} e^{-i\pi/6\,\sigma_3}$$

$$= \Psi_+(s)$$

Thus for $s \in \gamma_1$

$$\Psi_+(s) = \Psi_-(s) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Similarly (exercise!) for $s \in \gamma_3$ one finds

$$\Psi_+(s) = \Psi_-(s) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and for $s \in \gamma_2 \cup \gamma_4$, by definition

$$\Psi_+(s) = \Psi_-(s) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

(234.1) Thus $\quad \Psi_+(s) = \Psi_-(s)\, v^{\#}(s) \quad , \quad s \in \Sigma_\infty \setminus \{0\} = \bigcup_{i=1}^{4} \gamma_i$

Now for $z \in \mathbb{C} \setminus (-\infty, a]$, $z - a$ small

(234.2) $$\phi(z) = \frac{2}{3}(z-a)^{3/2}\, G(z)$$

where $G(z) = G(a) + (z-a)G'(a) + \cdots$ is analytic in a nbhood of a and $G(a) =$

(234.3) $$G(a) = t\, m\, (2a)^{\frac{1}{2}} h_1(a) > 0$$

and $(z-a)^{3/2} > 0$ for $z > a$.

To prove (234.2), write

$$\phi(z) = t\, m \int_a^z (s^2 - a^2)^{\frac{1}{2}} h_1(s)\, ds = t\, m \int_a^z (s-a)^{\frac{1}{2}} (s+a)^{\frac{1}{2}} h_1(s)\, ds$$

and expand $(s+a)^{\frac{1}{2}} h(s)$ in a power series in $(s-a)$, followed by term by term integration. To make sure that the branches in (234.2) agree, note that both sides in (234.2) are positive and agree for $a < z < a + \varepsilon$, $\varepsilon > 0$, and hence they agree for $z \in \mathbb{C} \setminus (-\infty, a]$, $z - a$ small by analytic continuation.

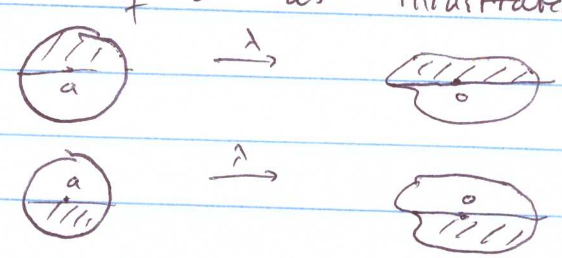
Now set

(235.1) $$\lambda(z) \equiv (z-a) \left(G(z)\right)^{2/3}$$

for $z - a$ small, where we choose the branch such that $\left(G(z)\right)^{2/3}$ is positive at $z = a$. Clearly

$\lambda(z)$ is analytic near $z = a$, $\lambda(a) = 0$ and $\lambda'(a) = (G(a))^{2/3} > 0$.

Hence $\lambda(z)$ maps a small neighborhood of a onto some open neighborhood of $\lambda = 0$. Further analysis (exercise: or see P.D. p217-8) shows that λ maps a neighborhood of a onto a neighborhood of 0 as illustrated in the following figures

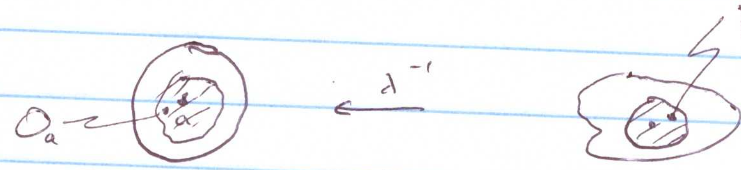


Choose and fix $\varepsilon > 0$ sufficiently small so that

$$D_\varepsilon = \{|\lambda| < \varepsilon\}$$

lies within the above neighborhood of $\lambda = 0$ and define the neighborhood O_a of a on p227 et seq by

(236.1) $$O_a = \lambda^{-1}(D_\varepsilon)$$



Note as $\lambda(z) < 0$ (resp. $\lambda(z) > 0$) for $z < a$ (resp $z > a$) we can define $(\lambda(z))^{3/2}$ as an analytic function in $O_a \setminus (-\infty, a]$ such that

(236.2) $$(\lambda(z))^{3/2} > 0 \quad \text{on } \{z > a\} \cap O_a$$

and hence

(236.3) $$(\lambda(z))^{3/2} = \frac{3}{2}\,\phi(z) \quad , \quad z \in O_a \setminus (-\infty, a].$$

The preceding calculations show that $\lambda^{-1}(\Sigma_\infty \cap D_\varepsilon)$ is an oriented contour in O_a arranged as follows:

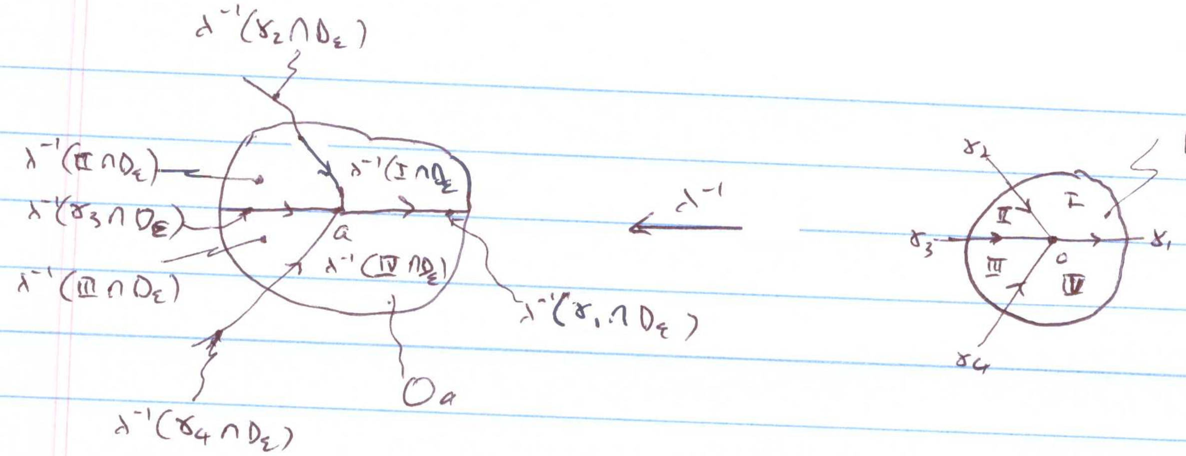


Fig 237.0

$$\mathrm{I}' \equiv \lambda^{-1}(\mathrm{I} \cap D_\varepsilon), \quad \mathrm{II}' \equiv \lambda^{-1}(\mathrm{II} \cap D_\varepsilon), \quad \mathrm{III}' \equiv \lambda^{-1}(\mathrm{III} \cap D_\varepsilon)$$

$$\mathrm{IV}' = \lambda^{-1}(\mathrm{IV} \cap D_\varepsilon)$$

Finally we are in a position to define a parametrix for the RHP $(\hat{W}_e, \hat{v}_e)$ in a nbhood of $z = a$ (recall from (223.0) that $\hat{v}_e = \hat{v}$ for $z \in \hat{\Sigma} \subset \hat{\Sigma}_e$). For $z \in O_a \setminus \lambda^{-1}(\Sigma_\infty \cap D_\varepsilon)$, set

(237.1) $$m_p(z) \equiv \Psi(n^{2/3} \lambda(z))$$

It follows from Figure (237.0) and the fact that Σ_∞ is composed of straight lines that

(237.2) $$n^{2/3} \lambda(\lambda^{-1}(\alpha \cap D_\varepsilon)) \subset \alpha, \quad \alpha = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$$

and

(237.2) $$n^{2/3} \lambda(\lambda^{-1}(\gamma_j \cap D_\varepsilon)) \subset \lambda_j, \quad j = 1, 2, 3, 4,$$

We conclude from (234.1) that

(238.1) $$m_{p+}(z) = m_{p-}(z)\; v^{\#}(z) \quad , z \in \lambda^{-1}(\Sigma_\infty \cap D_\varepsilon) \setminus \{a\}.$$

Now let $E(z)$ be any invertible analytic matrix in O_a and define

(238.2) $$\hat{W}_p(z) \equiv E(z)\, m_p(z)\, e^{n\phi(z)\sigma_3} \quad , z \in O_a \setminus \lambda^{-1}(\Sigma_\infty \cap D_\varepsilon)$$

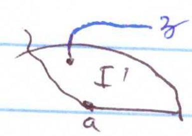
Clearly $\hat{W}_p(z)$ is analytic in O_a away from the lines λ_j, $j = 1,2,3,4$ and solves

(238.3) $$\hat{W}_{p+}(z) = \hat{W}_{p-}(z)\; \hat{v}(z)$$

across $\lambda^{-1}(\Sigma_\infty \cap D_\varepsilon) \setminus \{a\}$, by (230.0)

We now show it is possible to choose $E(z)$ such that $\hat{W}_p(z) = W_\infty(z)\left(I + O\left(\frac{1}{n}\right)\right)$ $\cancel{m_p(z) \sim W_\infty(z)}$ on ∂O_a. Thus $Z \equiv E(z)\, m_p(z)$ is the desired solution in (231.1).

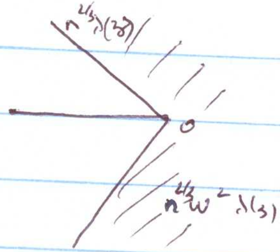
Consider $z \in I'$



Then $n^{2/3}\lambda(z) \in I$. From (232.2)

(238.4) $$\hat{W}_p(z) = E(z) \begin{pmatrix} Ai(n^{2/3}\lambda(z)) & Ai(\omega^2 n^{2/3}\lambda(z)) \\ Ai'(n^{2/3}\lambda(z)) & \omega^2 Ai'(\omega^2 n^{2/3}\lambda(z)) \end{pmatrix} e^{-i\frac{\pi}{6}\sigma_3}$$
$$\times\; e^{n\phi(z)\sigma_3}$$

Now for $z \in I'$, $n^{2/3}\lambda(z)$ and $\omega^2 n^{2/3}\lambda(z)$ both lie in a



closed sector away from the negative axis, as in the Figure above. Hence we may use (231.2) (231.3) to evaluate the Airy functions in (238.4) asymptotically as $n \to \infty$ for $z \in I' \cap \partial O_a$, since $|n^{2/3}\lambda(z)| = |\omega^2 n^{2/3}\lambda(z)| = n^{2/3}\varepsilon \to \infty$.

Writing $\hat{W}_p^n = E\, A(z)\, e^{-i\pi/6\,\sigma_3}\, e^{n\phi(z)\sigma_3}$ with

$$A(z) = (A_1 \quad A_2)$$

we find as $n \to \infty$

(239.1) $$A_1 = \begin{pmatrix} \frac{1}{2\sqrt{\pi}} \left(n^{2/3}\lambda(z)\right)^{-\frac14} e^{-\frac23 (n^{2/3}\lambda(z))^{3/2}} \left(1 + O(\tfrac1n)\right) \\ -\frac{1}{2\sqrt{\pi}} \left(n^{2/3}\lambda(z)\right)^{\frac14} e^{-\frac23 (n^{2/3}\lambda)^{3/2}} \left(1 + O(\tfrac1n)\right) \end{pmatrix}$$

and

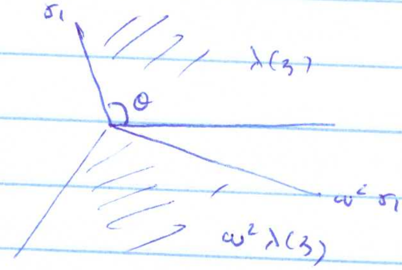
(239.2) $$A_2 = \begin{pmatrix} \frac{1}{2\sqrt{\pi}} \left(\omega^2 n^{2/3}\lambda\right)^{-\frac14} e^{-2/3 (\omega^2 n^{2/3}\lambda(z))^{3/2}} \left(1 + O(\tfrac1n)\right) \\ -\frac{1}{2\sqrt{\pi}} \left(\omega^2 n^{2/3}\lambda\right)^{\frac14} e^{-2/3 (\omega^2 n^{2/3}\lambda)^{3/2}} \left(1 + O(\tfrac1n)\right) \end{pmatrix}$$

The $1/4$ and $3/2$ roots are principal branches and so, in particular, by (236.3)

(239.3) $$(\lambda(z))^{3/2} = \frac32 \phi(z)$$

which implies $\quad e^{-\frac{2}{3}(n^{2/3}\lambda(z))^{3/2}} = e^{-n\phi(z)}$. On the other hand,

$\omega^2\lambda(z)$ lies in the sector below



$0<\theta<\frac{2\pi}{3}$

Hence

$$(\omega^2\lambda(z))^{3/2} = |\lambda(z)|^{3/2} e^{i[\arg\lambda(z) - 2\pi/3]3/2} =$$

(note $-\pi < \arg\lambda(z) - \frac{2\pi}{3} < \pi$).

$$= |\lambda(z)|^{3/2} e^{i\frac{3}{2}\arg\lambda(z)} e^{-i\pi}$$

$$= -(\lambda(z))^{3/2}$$

$$= -\frac{3}{2}\phi(z)$$

This implies

$$e^{-\frac{2}{3}(n^{2/3}\omega^2\lambda(z))^{3/2}} = e^{n\phi(z)}$$

As above

$$(\omega^2\lambda(z))^{\frac{1}{4}} = |\lambda(z)|^{\frac{1}{4}} e^{i(\arg\lambda(z) - \frac{2}{3}\pi)\frac{1}{4}}$$

$$= \lambda(z)^{\frac{1}{4}} e^{-i\pi/6}$$

Inserting these calculations into (239.1) and (239.2) we find

(240.1) $$\hat{w}_p(z) = E(z)\begin{pmatrix} \frac{1}{2\sqrt{\pi}}(n^{2/3}\lambda(z))^{-\frac{1}{4}}(1+O(\frac{1}{n})) & \frac{1}{2\sqrt{\pi}}(n^{2/3}\lambda(z))^{-\frac{1}{4}}e^{i\pi/6}(1+O(\frac{1}{n})) \\ -\frac{1}{2\sqrt{\pi}}(n^{2/3}\lambda(z))^{\frac{1}{4}}(1+O(\frac{1}{n})) & -\frac{\omega^2}{2\sqrt{\pi}}(n^{2/3}\lambda(z))^{\frac{1}{4}}e^{i\frac{\pi}{6}}(1+O(\frac{1}{n})) \end{pmatrix}$$

$$\times\ e^{-i\pi/6\,\sigma_3}.$$

$$= E(z)\,\frac{(n^{2/3}\lambda(z))^{-\sigma_3/4}}{2\sqrt{\pi}}\begin{pmatrix} e^{-i\pi/6} & e^{i\pi/3} \\ -e^{-i\pi/6} & -e^{i4\pi/3}\end{pmatrix}(I+O(\tfrac{1}{n}))$$

We choose

(241.1) $$E(z)\,\frac{\left(n^{2/3}\lambda(z)\right)^{-\sigma_3/4}}{2\sqrt{\pi}}\begin{pmatrix} e^{-i\pi/6} & e^{i\pi/3} \\ -e^{-i\pi/6} & -e^{i4\pi/3}\end{pmatrix} \equiv W_\infty(z)$$

where from p215

(241.2) $$W_\infty = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i\end{pmatrix}\left(\frac{z-a}{z+a}\right)^{\sigma_3/4}\begin{pmatrix} 1 & -i \\ 1 & i\end{pmatrix}$$

Now $$\begin{pmatrix} e^{-i\pi/6} & e^{i\pi/3} \\ -e^{-i\pi/6} & -e^{-i\pi4/3}\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} e^{-i\pi/6} & e^{i4\pi/3} \\ e^{-i\pi/6} & e^{i\pi/3}\end{pmatrix}$$

$$= e^{-i\pi/6}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i\end{pmatrix}$$

We require

$$E(z) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i\end{pmatrix}\left(\frac{z-a}{z+a}\right)^{\sigma_3/4}\begin{pmatrix} 1 & -i \\ 1 & i\end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i\end{pmatrix}^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} e^{i\pi/6}$$
$$2\sqrt{\pi}\left(n^{2/3}\lambda(z)\right)^{\sigma_3/4}$$

$$= \begin{pmatrix} 1 & 1 \\ i & -i\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\left(\left(\frac{z+a}{z-a}\right)\lambda(z)\right)^{\sigma_3/4}\cdot\sqrt{\pi}\; n^{\sigma_3/6}\, e^{i\pi/6}$$

Thus we choose for $z \in I'$

(241.3) $$E(z) \equiv \begin{pmatrix} 1 & -1 \\ -i & -i\end{pmatrix}\sqrt{\pi}\, e^{i\pi/6}\, n^{\sigma_3/6}\left((z+a)\left(G(z)\right)^{2/3}\right)^{\sigma_3/4}$$

as $\lambda(z) = (z-a)\left(G(z)\right)^{2/3}$ (see (235.1)).

Clearly $E(z)$ is analytic and invertible in $I' \subset O_a$. With this choice we thus find as $n \to \infty$

(241.4) $$\hat{W}_p(z) = W_\infty(z)\left(I + O\left(\tfrac{1}{n}\right)\right) \quad \text{for } z \in I' \cap \partial O_a.$$

Similar calculations (also see the abstract argument in P.D. p222 – 223) show that for the same $E(z)$

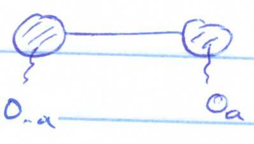
(242.1) $$\hat{W}_p(z) \equiv E(z)\, m_p(z)\, e^{n\phi(z)\sigma_3}$$

satisfies

$$\hat{W}_p(z) = W_\infty(z)\left(I + O\left(\tfrac{1}{n}\right)\right) \quad \text{for} \quad z \in \alpha' \cap \partial O_a$$

for all the sectors $\alpha' = I', II', III', IV'$. Thus

(242.2) $\hat{W}_p(z)$, together with the analogous construction in O_{-a}, is the desired parametrix in $(O_a \cup O_{-a}) \setminus \Sigma_e$ needed to complete the computation of the asymptotics of the OP's, as described in Lecture 14. As noted in (223.1), for z outside



$\hat{W}_p(z) \equiv W_\infty(z)$. $\square$.