⇒ Analogous calculations, which we leave as *exercises*, show that for $N \times N$ Hermitian matrices

$$(57.1) \qquad \left| \det \frac{\partial \vec{r}\vec{n}}{\partial(\Lambda, U)} \right| = \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^2 \; f_2(U), \qquad f_2(U) > 0$$

and for $2N \times 2N$ Hermitian self-dual matrix.

$$(57.2) \qquad \left| \det \frac{\partial \vec{r}\vec{n}}{\partial(\Lambda, U)} \right| = \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^4 \; f_4(U), \qquad f_4(U) > 0.$$

(see (2) (3) Chap 2)

For $(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$, the matrix

$$(57.3) \qquad V(\lambda) = \det\left(\lambda_j^{i-1}\right)_{1 \le i, j \le N}$$

is called the *Vandermonde* determinant. Explicit evaluation (*exercise*) shows that

$$V(\lambda) = \prod_{1 \le i < j \le N} (\lambda_j - \lambda_i) \tag{58.1}$$

Thus

real symmetric $\longrightarrow |V(\lambda)|^\beta = |V(\lambda)|^1$, $\beta = 1$

Hermitian $\longrightarrow |V(\lambda)|^\beta = |V(\lambda)|^2$, $\beta = 2$

Hermitian self-dual $\longrightarrow |V(\lambda)|^\beta = |V(\lambda)|^4$, $\beta = 4$

We see that the eigenvalues of Hermitian self-dual matrices experience the largest repulsion amongst themselves, while the eigenvalues of real symmetric matrices experience the least.

If we consider invariant convergence factors of the form (see p32)

$$F(M) = e^{-\operatorname{tr} Q(M)}$$

then locally $P_N(M)\,dM$ takes the form

(59.1)
$$P_N(M)\,dM = e^{-\sum Q(\lambda_i)} |V(\lambda)|^\beta f_\beta(U(p))\, d^N\lambda\, d^{N(N-1)/2} p$$

$\beta = 1, 2, 4$ respectively. Thus the eigenvalues and eigenvectors are independent.

We now show how to globalize this change of variables in order to compute probabilities of events.

Again we consider the case of real symmetric matrices S_N with distribution $P_N(M)\,dM$. Let $f : S_N \to \mathbb{R}$ be any bdd measurable function. We compute

(59.1) $\mathbb{E}xp(f) = \int f(M)\, P_N(M)\, dM$

For example, if $\chi_{a,b}(x)$ is the characteristic function of the interval (a,b) and

(59.2)
$$f(M) = \prod_{i=1}^{N} [1 - \chi_{a,b}(\lambda_i(M))].$$

where $\lambda_1(M), \ldots, \lambda_N(M)$ are the eigenvalues of M, then

$$\mathbb{E}\mathrm{xp}\,(f) = \int \prod_{i=1}^{N} \left(1 - \chi_{a,b}(\lambda_i(M))\right) P_N(M)\, dM$$

$$= \mathrm{Prob}\,(M : M \text{ has no eigenvalues in } (a,b))$$

This is the so-called <u>gap probability</u>

We now compute $\mathbb{E}\mathrm{xp}\, f$ for any f. (This form of the calculation is due to O. Conway.) As dM is invariant under conjugation by an orthogonal matrix

$$dM' = dM, \qquad M' = OMO^T$$

we have

(60.1) $$\int f(M)\, P_N(M)\, dM = \int f(OMO^T)\, P_N(OMO^T)\, dM$$

for any orthogonal matrix O. Let dO denote Haar measure on the orthogonal group O_N (Reference: Haar Measure, L. Nachbin). Integrating both sides of (60.1) wrt dO, we find

(60.2) $$\int f(M)\, P_N(M)\, dM = \int \left(\int f(OMO^T)\, P_N(OMO^T)\, dO \right) dM$$

As $\text{meas}(S_N \setminus A_N) = 0$, we have

$$\int f(M)\, P_N(M)\, dM = \int_{A_N} \left(\int_{O_N} f(OMO^T)\, P_N(OMO^T)\, dO \right) dM$$

Now for $M \in A_N$, the eigenvalues are distinct

$$\Lambda(M) = \text{diag}(\lambda_1(M), \ldots, \lambda_N(M)),$$

$$\lambda_1(M) < \cdots < \lambda_N(M),$$

and the $\boxed{\cancel{\text{first components } u_1(1,M), \ldots, u_N(1,M) \text{ of}}}$ the eigenvector matrix $O(M) = (u_1(M), \ldots, u_N(M))$ is uniquely specified as a smooth matrix by the condition

$$u_j(1,M) > 0, \quad j = 1, \ldots, N.$$

Thus for $M \in A_N$,

$$\int_{O_N} f(OMO^T)\, P_N(OMO^T)\, dO$$

$$= \int_{O_N} f\left(O O(M) \Lambda(M) (O O(M))^T \right) P_N\left(O O(M) \Lambda(M) (O O(M))^T \right) dO$$

$$= \int_{O_N} f(O \Lambda(M) O^T)\, P_N(O \Lambda(M) O^T)\, dO$$

as Haar measure is invariant under right (and left) multiplication. These calculations show that

(62.1) $$\mathbb{E}xp(f) = \int_{A_N} \left[\int_{O_N} f(O\Lambda(M)O^T)\, P_N(O\Lambda(M)O^T)\, dO \right] dM$$

Note that $\int_{O_N} f(O\Lambda(M)O^T)\, P_N(O\Lambda(M)O^T)\, dO$ depends

<u>only</u> on the eigenvalues of M.

Let $g : \mathbb{R}^N \to \mathbb{R}$ be an arbitrary bded mble

function and consider $g(\Lambda(M)) = g(\lambda_1(M), \ldots, \lambda_N(M))$.
Assume $\int |g(\Lambda(M))|\, dM < \infty$.
We compute

$$\int_{A_N} g(\Lambda(M))\, dM$$

Now for each $M \in A_N$, $\exists$ an open nbhood O_M

of M, $M \in O_M \subset A_N$ with a smooth parameterization

(62.2) $$O_M = \left\{ M(\Lambda, p) = O(p)\, \Lambda\, O(p)^T, \quad \lambda_1 < \cdots < \lambda_N, \quad p^2 = \sum_{i=1}^{\ell} p_i^2 < \varepsilon^2 \right\}$$

where $O(p=0) = O(M)$. The sets $\{O_M : M \in A_N\}$

provide an open cover for A_N. Hence $\exists$ a (countable) partition

of unity $\{g_j\}_{j=1}^{\infty}$ subordinate to $\{O_M\}$ i.e. the g_j's are

Now from the calculation above O_1^+, the set of orthogonal matrices $\{O\}$ with $o_j(1) = (Oe_j, e_1) > 0$, $j = 1, \ldots, N$ is an open subset of the orthogonal group. Each $O \in O_1^+$ is contained in an open nbhood V_O

(63.1) $$O \in V_O = \{ O(p) : p^2 = \sum_{j=1}^{N} p_j^2 < \varepsilon^2 \} \subset O_1^+$$

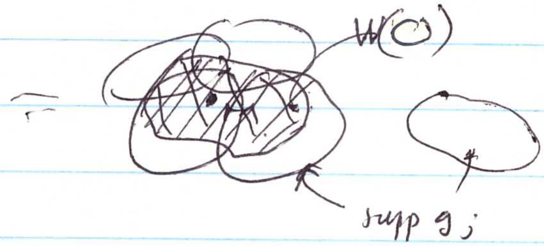
(63.2) $$O(0) = O$$

where $O(p)$ is a smooth function of p. The sets $\{V_O\}$ provide an open cover for O_1^+.

It follows that $\exists$ a (countable) partition of unity $\{g_j\}_{j=1}^{\infty}$ subordinate to $\{V_O : O \in O_1^+\}$ i.e. the g_j's are C^∞ functions on O_1^+ such that:

$$\sum$$

(64.1) The collection of supports $\{\mathrm{supp}\, g_j : j \geq 1\}$ is <u>locally finite</u> i.e. for any $O \in O_1^+$, $\exists$ a nbhd $W(O)$ of O st $W(O) \cap \{\mathrm{supp}\, g_j\} \neq \phi$ for only a <u>finite</u> # of j's.



(64.2) $\sum_j g_j(O) = 1 \quad \forall O \in O_1^+$ and $g_j(O) \geq 0 \quad \forall O \in O_1^+$ and $j \geq 1$.

(64.3) For any j, $\exists$ V_O st $\mathrm{supp}\, g_j \subset V_O$

In addition

(64.4) $\mathrm{supp}\, g_j$ is compact for each $j \geq 1$

The existence of the partition of unity $\{g_j\}$ subordinate to the open cover, with the additional property (64.4) follows from the general theory of manifolds (see e.g

F.W. Warner, Foundations of Differentiable Manifolds and Lie Groups, 1971, Th^m 1.11 p10.) Note that as O_1^+ is an open subset of the orthogonal group, it is indeed a manifold.

Again from the calculations above, every $M \in A_N$ has a unique orthogonal matrix $O(M) \in O_1^+$ st $M = O(M)\,\Lambda(M)\,O(M)^T$ and $M \mapsto O(M)$ is smooth. Thus

(65.1) $$\int_{A_N} g(\Lambda(M))\,dM = \int_{A_N} \sum_{j=1}^{\infty} g_j(O(M)).\, g(\Lambda(M))\,dM$$

$$= \sum_{j=1}^{\infty} \int_{A_N} g_j(O(M))\, g(\Lambda(M))\,dM.$$

Now fix j: then $\operatorname{supp} g_j \subset V_{\tilde{O}}$ for some $\tilde{O} \in O_1^+$

$V_{\tilde{O}} = \{\tilde{O}(p) : p^2 < \varepsilon^2\}$.

supp g_j

Consider the open set

$$A_{\tilde{O}} = \{M = O\Lambda O^T : O \in V_{\tilde{O}},\ \lambda_1 < \lambda_2 < \dots < \lambda_N\}$$

in A_N.

Now $g_j(O(M)) > 0 \Rightarrow O(M) \in V_{\tilde{\delta}}$.

Hence $M = O(M)\Lambda(M)O(M)^T \in A_{\tilde{\delta}}$

$\boxed{\cancel{= \tilde{O}(P)\Lambda(M)\tilde{O}(P)^T \in \tilde{\mathcal{O}}_\varepsilon}}$ and we have

$$\int_{A_N} g_j(O(M))\, g(\Lambda(M))\, dM$$

$$= \int_{A_{\tilde{\delta}}} g_j(O(M))\, g(\Lambda(M))\, dM$$

and we are reduced to the local calculation in the previous lecture. Hence

(66.1) $$\int_{A_N} g_j(O(M))\, g(\Lambda(M))\, dM$$

$$= \int_{\lambda_1<\cdots<\lambda_N}\!\!\cdots\int \int_{P^2<\varepsilon^2} \Big(\prod_{1\le i<k\le N} |\lambda_i-\lambda_k|\Big) g_j(\tilde{O}(P))\, g(\lambda)\, f_1^{(j)}(P)\, dP\, d\lambda^N$$

$$= \Big(\int_{\lambda_1<\cdots<\lambda_N}\!\!\cdots\int g(\lambda) \prod_{1\le i<k\le N} |\lambda_i-\lambda_k|\, d\lambda^N\Big) \int_{P^2<\varepsilon^2} f_1^{(j)}(P)\, g_j(\tilde{O}(P))\, dP$$

Here $\varepsilon = \varepsilon_j$: why?

Inserting this relation into (65.1) we find

(66.2) $$\int_{A_N} g(\Lambda(M))\, dM = \Big(\int_{\lambda_1<\cdots<\lambda_N}\!\!\cdots\int g(\lambda) \prod_{i<k} |\lambda_i-\lambda_j|\, d\lambda^N\Big)\, C$$

where

(67.1) $$C = \sum_{j=1}^{\infty} \int_{p^2 < \varepsilon_j^2} f_1^{(j)}(p)\, g_j(\tilde{O}(p))\, dp$$

is indep. of g

We may now apply the above calculation to $\mathbb{E}\mathrm{xp}\, f$ in (62.1). We find

(67.2) $$\mathbb{E}\mathrm{xp}\, f = C \int_{\lambda_1 < \cdots < \lambda_N} \cdots \int \Big[\int_{O_N} f(O \Lambda O^T)\, P_N(O \Lambda O^T)\, dO \Big] \times \prod_{1 \le i < k \le N} |\lambda_i - \lambda_k|\, d^N\lambda$$

Setting $f = 1$ in (67.2) we find

$$1 = C \int_{\lambda_1 < \cdots < \lambda_N} \cdots \int \Big(\int_{O_N} P_N(O \Lambda O^T)\, dO \Big) \times \prod_{1 \le i < k \le N} |\lambda_i - \lambda_k|\, d^N\lambda$$

Thus where O_N denotes the orthogonal group,

(67.3) $$\mathbb{E}\mathrm{xp}\, f = \frac{\displaystyle\int_{\lambda_1 < \cdots < \lambda_N} \cdots \int \int_{O_N} f(O \Lambda O^T) \prod_{i<k} |\lambda_i - \lambda_k|\, P_N(O \Lambda O^T)\, d^N\lambda\, dO}{\displaystyle\int_{\lambda_1 < \cdots < \lambda_N} \cdots \int \int_{O_N} \prod_{i<k} |\lambda_i - \lambda_k|\, P_N(O \Lambda O^T)\, d^N\lambda\, dO}$$

In particular if

- $P_N(M)$ is invariant under O_N, e.g.,

$$P_N(M) = e^{-\operatorname{tr} Q(M)} = P_N(\lambda)$$

- if $f(M)$ is also O_N invariant

$$f(M) = f(\lambda_1(M), \ldots, \lambda_N(M)) = f(\lambda)$$

eg if $f(M)$ is given as in (54.2)

then

(68.1)
$$\mathbb{E}\mathrm{xp}\, f = \frac{\int_{\lambda_1<\cdots<\lambda_N} \int_{O_N} f(\lambda) \prod_{i<k} |\lambda_i - \lambda_k| \, P_N(\lambda)\, d^N\lambda}{\int \cdots \int_{\lambda_1<\cdots<\lambda_N} \int_{O_N} \prod_{i<k} |\lambda_i - \lambda_k| \, P_N(\lambda)\, d^N\lambda}$$

$$= \frac{\int \cdots \int_{\lambda_1<\cdots<\lambda_N} f(\lambda) \prod_{i<k} |\lambda_i - \lambda_k| \, P_N(\lambda)\, d^N(\lambda)}{\int \cdots \int_{\lambda_1<\cdots<\lambda_N} \prod_{i<k} |\lambda_i - \lambda_k| \, P_N(\lambda)\, d^N\lambda}$$

Thus the eigenvectors integrate out completely when we compute the statistics of the eigenvalues <u>in the case of invariant ensembles</u>. In the case of general Wigner ensembles, this is not the case and the change of variables $M = O\Lambda O^T \to (\Lambda, O)$ does not simplify the computation of the statistics of the eigenvalues. One needs to take a different approach (see ref 10 L. Erdős)

<u>Note</u>: From (67.3) we see that when it comes to computing $\mathbb{E}xp\, f$ for any $f = f(M)$, integrating wrt to $P_N(M)\,dM$ is the same as integrating $f(O\Lambda O^T)$ wrt

$$P_N(O\Lambda O^T) \prod_{1\le i<k\le N} |\lambda_i - \lambda_k| \, d^N\lambda \, dO \qquad \text{over } \{\lambda_1 < \cdots < \lambda_N\} \times O_N$$

This is $\boxed{\text{not}}$ to say that dM is the product of $\prod_{i<k} |\lambda_i - \lambda_k| d^N\lambda$ and ~~Haar~~ measure, i.e.

$$dM = \prod_{i<k} |\lambda_i - \lambda_k| \, d^N\lambda \, dO \quad , \; dO = \text{Haar measure.}$$

What we are saying is that if we integrate functions of the special form $f(M) = f(O\Lambda O^T)$ with respect to these measures, we obtain the same result. Said differently, what (67.3) in fact tells us is that $P_N(M)\,dM$ is the $\boxed{\text{push forward } \mu_\varphi}$ of the measure

$$d\mu(\Lambda, O) = \Big(\prod_{i<k} |\lambda_i - \lambda_k|\Big) P_N(O\Lambda O^T) \, d^N\lambda \, dO \Big/ \int\!\cdots\!\int_{\lambda_1 \le \cdots \le \lambda_N} \int_{O_N} \prod_{i<k} |\lambda_i - \lambda_k| P_N(O\Lambda O^T) \, d^N\lambda \, dO$$

under the map $\varphi : (\Lambda, O) \mapsto M = O\Lambda O^T$ from $\{\lambda_1 \le \cdots \le \lambda_N\} \times O_N$ onto S_N,

$$\int f(M) P_N(M) \, dM = \int f(\varphi(\Lambda, O)) \, d\mu(\Lambda, O) = \int f(M) \, d\mu_\varphi(M)$$

A similar argument for the Hermitian ensemble, $(\beta=2)$ leads to exactly the same formulae for $\mathbb{E}xp\, f$ as above except $\prod_{1\le i<k\le N} |\lambda_i - \lambda_k|$ is replaced by $\prod_{1\le i<k\le N} |\lambda_i - \lambda_k|^2$. And for the Hermitian self-dual case $(\beta=4)$ $\prod_{i<k} |\lambda_i - \lambda_k|$ is replaced by $\prod_{1\le i<k\le N} (\lambda_i - \lambda_k)^4$

We leave the details as an exercise.

We now show how to compute $\mathbb{E}xp\, f$ for invariant Hermitian ensembles $(\beta=2)$

(70.1) $$P_N(M) = e^{-\operatorname{tr} Q(M)}, \quad Q(x) \to +\infty \text{ as } |x| \to \infty$$

and f is of the form

(70.2) $$f(M) = \prod_{i=1}^{N} \left[1 - g(\lambda_i(M))\right]$$

for some bounded function $g: \mathbb{R} \to \mathbb{R}$.

We follow the method of Tracy & Widom (see (2)(3)).

We will need some auxiliary results from functional analysis and matrix theory.

Let X and Y be 2 separable Hilbert spaces.

Let $A : X \to Y$ be trace class (see eg B. Simon, Trace ideals and their applications) and let $B : Y \to X$ be bounded. Then

(71.1) $$\det(1_X + BA) = \det(1_Y + AB)$$

Let $\{f_j(x)\}$ and $\{g_j(x)\}$ denote real- or complex valued functions on $\mathbb{R}$ that are in $L^2(d\mu)$ for some Borel measure $d\mu$ on $\mathbb{R}$

Then the following identity is true

(71.2) $$\int \cdots \int \det(f_j(x_k))_{j,k=1,\dots,N} \det(g_j(x_k))_{j,k=1,\dots,N} d\mu(x_1) \cdots d\mu(x_N)$$
$$= N! \det\left(\int f_j(x) g_k(x) d\mu(x)\right)_{j,k=1,\dots,N}.$$

<u>Proof of (71.1)</u> Note first that because $A : X \to Y$ is trace class, it can be represented (see [Simon 7]) with

form

$$A = \sum_{i=1}^{\infty} \sigma_i (u_i, \cdot)\, v_i$$

i.e. $Aw = \sum_1^\infty \sigma_i (u_i, w) v_i$

where the <u>singular values</u> σ_i, $i \geq 1$ are positive and summable, $\sum \sigma_i < \infty$, and the $\{u_i\}$ and $\{v_i\}$ are orthogonal & normalized in X and Y resp. i.e.

$$(u_i, u_j)_X = \delta_{ij} \quad , \quad (v_i, v_j) = \delta_{ij}$$

Extending $\{v_i\}$ if necessary to an orthonormal basis $\{v_i\} \cup \{v_i'\}$ of Y, we have

$$\begin{aligned}
\mathrm{tr}_Y AB &= \sum_{k=1}^{\infty} (v_k, AB v_k) + \sum_{k=1}^{\infty} (v_k', AB v_k') \\
&= \sum_{k=1}^{\infty} \Big(v_k, \sum_{i=1}^{\infty} \sigma_i (B^* u_i, v_k) v_i\Big) + 0 \\
&= \sum_{i=1}^{\infty} \sigma_i (B^* u_i, v_i) \\
&= \sum_{i=1}^{\infty} \sigma_i (u_i, B v_i)
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathrm{tr}_X BA &= \sum_{k=1}^{\infty} \Big(u_k, \sum_{i=1}^{\infty} \sigma_i (u_i, u_k) B v_i\Big) \\
&= \sum_{k=1}^{\infty} (u_k, \sigma_k B v_k) = \mathrm{tr}_Y AB .
\end{aligned}$$

(73.1) ie $\quad tr_X BA = tr_Y AB$

Since $T \mapsto \det(1+T)$ is analytic on the trace class operators, it is enough to show that

$$\det(1_Y + \alpha AB) = \det(1_X + \alpha BA)$$

for α small. But then by the formula (see [Simon])

$$\det(1+T) = e^{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} tr\, T^k}, \quad T \text{ small}$$

it is enough to show $\quad tr_X (BA)^k = tr_Y (AB)^k, \quad k \geq 1.$

But by (73.1) for $k \geq 1$,

$$
\begin{aligned}
tr_X (BA)^k &= tr_X [(BA)^{k-1} B] A \\
&= tr_Y A [(BA)^{k-1} B] \\
&= tr_Y (AB)^k
\end{aligned}
$$

as desired.

<u>Remarks</u>

Formula (71.1), the commutation formula is extremely useful in many, many different areas of mathematics

For example, if we replace B by $-B/\lambda$, $\lambda \neq 0$ we see from (71.1) that

$$\det(1_X - \tfrac{1}{\lambda}BA) = 0 \iff \det(1_Y - \tfrac{1}{\lambda}AB) = 0$$

Thus

(74.1) $$\operatorname{spec} BA \setminus 0 = \operatorname{spec} AB \setminus 0$$

The map $AB \to BA$ is the fundamental isospectral action (see Deift, DMJour, 1978, Applications of a Commutation Formula, for many different applications of (71.1). Some of the most applications of the formula involve the observation that X and Y are in general different spaces, so one may be finite dimensional, the other infinite dimensional; In particular this is the situation, as we will see, in RMT.