More generally let X and Y be Banach spaces and let $A: X \to Y$, $B: Y \to X$ be bdd linear operators. Then the following <u>commutation formula</u> is true:

(75.1) $$\frac{\lambda}{\lambda + AB} + A \frac{1}{\lambda + BA} B = 1_Y$$

in the sense that if $0 \neq -\lambda \in \rho(BA)$ = resolvent set of BA, then $-\lambda \in \rho(AB)$ and

$$\frac{1}{\lambda}\left(1_Y - A \frac{1}{\lambda + BA} B\right)$$

is the resolvent of AB, and vice versa. In particular we see that (74.1) is true for general bounded operators in Banach spaces. (75.1) is also true, suitably interpreted, for certain classes of unbounded operators (see Deift [DMJ, 1978]).

We now prove (71.2). Let μ and $f_i, g_j \in L^2(d\mu)$ be given. Then

$$\int \cdots \int \det(f_j(x_k))_{j,k=1,\dots,N} \det(g_j(x_k))_{j,k=1,\dots,N} \, d\mu(x_1)\cdots d\mu(x_N)$$

$$= \sum_{\sigma,\tau} \operatorname{sgn}\sigma \operatorname{sgn}\tau \int f_{\sigma(1)}(x_1)\cdots f_{\sigma(N)}(x_N) \times g_{\tau(1)}(x_1)\cdots g_{\tau(N)}(x_N)\, d\mu(x_1)\cdots d\mu(x_N)$$

$$= \sum_{\sigma,\tau} \operatorname{sgn}\sigma \operatorname{sgn}\tau \left[\int f_{\sigma(1)}(x_1) g_{\tau(1)}(x_1)\, d\mu(x_1)\right]\cdots\left[\int f_{\sigma(N)}(x_N) g_{\tau(N)}(x_N)\, d\mu(x_N)\right]$$

$$= \sum_{\sigma,\tau} \operatorname{sgn}\sigma\circ\tau^{-1} \left[\int f_{\sigma\circ\tau^{-1}(\tau(1))}(x_1)\, g_{\tau(1)}(x_1)\, d\mu(x_1)\right]$$

$$\times \cdots \left[\int f_{\sigma\circ\tau^{-1}(\tau(N))}(x_N)\, g_{\tau(N)}\, d\mu(x_N)\right].$$

$$= \sum_{\sigma} \sum_{\sigma\circ\tau^{-1}:\, \tau\in S_N} \operatorname{sgn}\sigma\circ\tau^{-1} \left[\int f_{\sigma\circ\tau^{-1}(\tau(1))}(x)\, g_{\tau(1)}(x)\, d\mu(x)\right]$$

$$\times\cdots\times \left[\int f_{\sigma\circ\tau^{-1}(\tau(N))}(x)\, g_{\tau(N)}(x)\, d\mu(x)\right]$$

$$= \sum_{\sigma} \sum_{\sigma\circ\tau^{-1}:\, \tau\in S_N} \operatorname{sgn}\sigma\circ\tau^{-1} \left[\int f_{\sigma\circ\tau^{-1}(1)}(x)\, g_1(x)\, d\mu(x)\right]$$

$$\times\cdots\times \left[\int f_{\sigma\circ\tau^{-1}(N)}(x)\, g_N(x)\, d\mu(x)\right]$$

$$= \sum_{\sigma} \sum_{\tau'\in S_N} \operatorname{sgn}\tau' \left[\int f_{\tau'(1)}(x)\, g_1(x)\, d\mu(x)\right]\cdots\left[\int f_{\tau'(N)}(x)\, g_N(x)\, d\mu(x)\right]$$

$$= \sum_{\sigma} \det \left(\int f_j(x)\, g_k(x)\, d\mu(x) \right)_{j,k=1,\dots,N}$$

$$= N! \det \left(\int f_j(x)\, g_k(x)\, d\mu(x) \right)_{j,k=1,\dots,N}$$

as desired.

The particular case

$$d\mu(x) = \sum_{k=1}^{M} \delta_{\lambda_k}(x)$$

for a given set of numbers $\lambda_1, \dots, \lambda_M$. Then the LHS of (71.2) becomes

$$\sum_{k_1=1}^{M} \cdots \sum_{k_N=1}^{M} \int \cdots \int \det \begin{pmatrix} f_1(x_1) & \cdots & f_1(x_N) \\ \vdots & & \\ f_N(x_1) & \cdots & f_N(x_N) \end{pmatrix} \det \begin{pmatrix} g_1(x_1) & \cdots & g_1(x_N) \\ \vdots & & \\ g_N(x_1) & \cdots & g_N(x_N) \end{pmatrix}$$

$$\delta_{\lambda_{k_1}}(x_1) \cdots \delta_{\lambda_{k_N}}(x_N)$$

$$= \sum_{k_1=1}^{M} \cdots \sum_{k_N=1}^{M} \det \begin{pmatrix} f_1(\lambda_{k_1}) & \cdots & f_1(\lambda_{k_N}) \\ \cdots & & \\ f_N(\lambda_{k_1}) & & f_N(\lambda_{k_N}) \end{pmatrix} \det \begin{pmatrix} g_1(\lambda_{k_1}) & \cdots & g_1(\lambda_{k_N}) \\ \vdots & & \vdots \\ g_N(\lambda_{k_1}) & \cdots & g_N(\lambda_{k_N}) \end{pmatrix}$$

$$= N! \sum_{1 \le k_1 < k_2 < \cdots < k_N \le M} \det \begin{pmatrix} f_1(\lambda_{k_1}) & \cdots & f_1(\lambda_{k_N}) \\ \vdots & & \\ f_N(\lambda_{k_1}) & \cdots & f_N(\lambda_{k_N}) \end{pmatrix}$$

$$\times \det \begin{pmatrix} g_1(\lambda_{k_1}) & \cdots & g_N(\lambda_{k_1}) \\ \vdots & & \vdots \\ g_1(\lambda_{k_N}) & \cdots & g_N(\lambda_{k_N}) \end{pmatrix}$$

On the other hand, the RHS of (71.2) gives

$$N! \det \left(\sum_{i=1}^{M} f_j(\lambda_i)\, g_k(\lambda_i) \right)_{j,k=1,\dots,N}.$$

Equating these expressions we see that

if $$F = (F_{ij}) = \{f_i(\lambda_j)\} = \begin{pmatrix} f_1(\lambda_1) & \cdots & f_1(\lambda_M) \\ \vdots & & \\ f_N(\lambda_1) & \cdots & f_N(\lambda_M) \end{pmatrix}$$

$= N \times M$

and $$G = (G_{ij}) = (g_j(\lambda_i)) = \begin{pmatrix} g_1(\lambda_1) & \cdots & g_N(\lambda_1) \\ \vdots & & \vdots \\ g_1(\lambda_M) & \cdots & g_N(\lambda_M) \end{pmatrix}$$

$= M \times N$

we obtain the classical <u>Cauchy–Binet formula</u>

(79.1) $$\det FG = \sum_{1 \le j_1 < \cdots < j_N \le M} \det \begin{pmatrix} F_{1j_1} & F_{1j_2} & \cdots & F_{1j_N} \\ \vdots & & & \vdots \\ F_{Nj_1} & F_{Nj_2} & \cdots & F_{Nj_N} \end{pmatrix} \det \begin{pmatrix} G_{j_1 1} & \cdots & G_{j_1 N} \\ \vdots & & \vdots \\ G_{j_N 1} & \cdots & G_{j_N N} \end{pmatrix}$$

eg if $F = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$, $G = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{pmatrix}$

then

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{pmatrix}$$

$$= \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \det \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} + \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} \det \begin{pmatrix} c_1 & d_1 \\ c_3 & d_3 \end{pmatrix}$$

$$+ \det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} \det \begin{pmatrix} c_2 & d_2 \\ c_3 & d_3 \end{pmatrix}$$

Of course if $N = M$, (79.1) is just the familiar fact that

$$\det FG = \det F \det G$$

and for $N > M$ both LHS and RHS are 0 (why?)

We will also need the following <u>basic definition</u>

Let $d\mu(x)$ be a Borel measure on $\mathbb{R}$ with all moments finite,

(80.1) $$\int_{\mathbb{R}} |x|^m \, d\mu(x) < \infty , \qquad m = 0, 1, 2, \ldots$$

Suppose that the support of $d\mu$ is infinite i.e. $d\mu$ is not a finite linear combination of delta functions. Then by the Gram-Schmidt procedure there exist unique monic polynomials

$$\pi_k(x) = x^k + \cdots , \qquad k \geq 0$$

which are orthogonal with respect to $d\mu$ i.e.

$$\int_{\mathbb{R}} \pi_k(x) \, \pi_j(x) \, d\mu(x) = 0 \qquad \text{for } j \neq k .$$

For

$$\gamma_k = \left(\int_{\mathbb{R}} \pi_k^2(x) \, d\mu(x) \right)^{-\frac{1}{2}} > 0 , \quad k = 0, 1, 2, \ldots$$

set

$$p_k(x) = \gamma_k \pi_k(x) \quad , \quad k \geq 0 .$$

The polynomials $\{p_k\}$ are the orthonormal polynomials with respect to $d\mu$, i.e.

$$\int_{\mathbb{R}} p_k(x)\, p_j(x)\, d\mu(x) = \delta_{kj} \quad , \quad j,k = 0,1,2,\ldots$$

The p_k's are called the orthonormal polynomials (wrt $d\mu$) and the π_k's are called the monic orthogonal polynomials (wrt $d\mu$). Exercise: what happens if $d\mu$ has finite support?

As we will see, the p_k's and π_k's play a central role in RMT.

We now show how to compute certain basic statistics for invariant unitary ensembles with probability distributions

$$P_N(M)\, dM = \frac{e^{-\operatorname{tr} Q(M)}\, dM}{Z_N}$$

where $Q(x) \to +\infty$ sufficiently rapidly (see pp 84, 85 below) as $|x| \to \infty$.

In particular we will compute

$$\langle f \rangle = \int f(M) P_N(M)\, dM$$

for $f(M)$ of the form

(82.0) $$f(M) = \det(I + g(M))$$

for bounded functions $g : \mathbb{R} \to \mathbb{R}$.

From the Hermitian analog of (68.1) we have

(82.1) $$\begin{aligned}\langle f \rangle &= C_N' \int_{\lambda_1 < \cdots < \lambda_N} \prod_{i=1}^{N} (1 + g(\lambda_i)) \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{N} w(\lambda_i)\, d^N\lambda \\ &= C_N' \int_{\lambda_1 < \cdots < \lambda_N} \prod_{i=1}^{N} (1 + g(\lambda_i))\, V^2(\lambda) \prod_{i=1}^{N} w(\lambda_i)\, d^N\lambda\end{aligned}$$

where

(82.2) $$w(x) = e^{-Q(x)}$$

(82.3) $$V(\lambda) = \text{vandermonde} = \det(\lambda_i^{j-1})_{1 \le i, j \le N}.$$

and

(82.4) $$C_N' = \left(\int_{\lambda_1 < \cdots < \lambda_N} V^2(\lambda) \prod_1^N w(\lambda_i)\, d^N\lambda \right)^{-1}$$

As the integrands in (82.1) and (82.4) are invariant under all permutation σ

$$(\lambda_1, \ldots, \lambda_N) \to (\lambda_{\sigma 1}, \ldots, \lambda_{\sigma N})$$

we may rewrite (82.1) (82.4) in the following symmetrized form

(83.1) $$\langle f \rangle = C_N \int_{\mathbb{R}^N} \prod_{i=1}^{N} (1+g(\lambda_i)) \, V^2(\lambda) \prod_i^N w(\lambda_i) \, d^N\lambda$$

where

(83.2) $$C_N = C_N' (N!)^{-1}$$

Now define

(83.3) $$f_i(x) = g_i(x) = x^{i-1}, \quad 1 \le i \le N$$

Then

$$V(\lambda)^2 = \det(f_i(\lambda_j))_{1\le i,j\le N} \det(g_i(\lambda_j))_{1\le i,j\le N}$$

Hence we can apply the generalized Cauchy-Binet formula to (83.1) with measure $d\mu(x) \equiv w(x)(1+g(x))$

(the fact that $d\mu$ may be a signed measure, as opposed to a pos. measure, clearly does not affect the proof of (83.11). We obtain

$$\langle f \rangle = C_N' \det \left(\int f_j(\lambda) g_k(\lambda) \, d\mu(\lambda) \right)_{j,k=1}^{N}$$

where we note from (82.4) that C_N' depends <u>only</u> on <u>N and w</u>, but <u>not on $g(x)$</u>.

We have

(84.1) $$\int f_j(\lambda) g_k(\lambda) \, d\mu(\lambda) = \int f_j(\lambda) g_k(\lambda) (1+g(\lambda)) w(\lambda) \, d\lambda .$$
$$= \int \lambda^{j+k-2} (1+g(\lambda)) w(\lambda) \, d\lambda$$

Let $p_j(\lambda)$, $j \geq 0$, be the orthogonal polynomials with respect to the measure $w(\lambda) d\lambda$ i.e.

$$\int_{\mathbb{R}} p_k(\lambda) p_j(\lambda) w(\lambda) d\lambda = \delta_{jk} , \quad j,k \geq 0 .$$

(<u>Note</u> that $w(\lambda) d\lambda = e^{-Q(\lambda)} d\lambda$ does not have finite support: also by "sufficient decay" we are assuming, at least, that $e^{-Q(\lambda)} d\lambda$ has finite moments: not that this implies

$$\int |V(\lambda)|^2 \prod_{i=1}^{N} w(\lambda_i) \, d^N\lambda < \infty$$

so that $P_N(M)\,dM$ is a normalized prob. measure with finite moments).

Now as we can add rows and colums without changing a determinant, we have (recall $p_j(\lambda) = \gamma_j \pi_j(\lambda)$)

(85.1) $$\langle f \rangle = C_N' \det\left(\int \pi_j(\lambda)\, \pi_k(\lambda)\, (1+g(\lambda))\, w(\lambda)\, d\lambda\right)_{0\le j,k\le N-1}$$

$$= \frac{C_N'}{\prod_{j=0}^{N-1} \gamma_j^2} \det\left(\int p_j(\lambda)\, p_k(\lambda)\, (1+g(\lambda))\, w(\lambda)\, d\lambda\right)_{0\le j,k\le N-1}$$

Set

(85.2) $$\phi_j(x) = p_j(x)\, w(x)^{\frac{1}{2}}, \quad j \ge 0$$

Have $\int \phi_j(x)\, \phi_k(x)\, dx = \delta_{jk}$, $j,k \ge 0$.

Then (85.1) takes the form

(85.3) $$\langle f \rangle = \frac{C_N'}{\prod_{j=0}^{N-1} \gamma_j^2} \det\left(\int \phi_j(\lambda)\, \phi_k(\lambda)\, (1+g(\lambda))\, d\lambda\right)_{0\le j,k\le N-1}$$

$$= C_N'' \det\left(\delta_{j,k} + \int g(\lambda)\, \phi_j(\lambda)\, \phi_k(\lambda)\, d\lambda\right)_{j,k=0}^{N-1}$$

where again C_N'' does not depend on $g(\lambda)$. Setting $g=0$, we have $f(M)=1$ and so LHS of (85.3) $=1$

But RHS $= C_N'' \times \det(\delta_{j,k}) = C_N''$, so $C_N''=1$.

Thus

(86.1) $$\langle f\rangle = \det\left(\delta_{j,k} + \int_{\mathbb{R}} \phi_j(x)\,\phi_k(x)\,g(x)\,dx\right)_{j,k=0}^{N-1}.$$

Now we are primarily interested in the situation where the size N of the matrices become large. We see from (86.1) that $\langle f\rangle$ is expressed in terms of determinants of larger and larger size. Limits of this kind are generally very difficult to control. Fortunately we can use (71.1) $\det(1_X + AB) = \det(1_Y + BA)$ to reduce (86.1) to the (Fredholm) determinant on a *fixed* space (see below). Such limits are, generally speaking, easier to control; in particular if K_N is a trace class operator on a (fixed) Hilbert space $\mathcal{H}$, and $K_N \to K$ in trace norm, then

$$\det(1 + K_N) \to \det(1+K)$$

It is just a matter of continuity of the determinant in the trace norm.

Now let $A : L^2(\mathbb{R}) \to \mathbb{C}^N$ denote the bounded operator

(87.1) $$(Ah)_j = \int \phi_j(x)\, g(x)\, h(x)\, dx \quad , \quad j = 0, \ldots, N-1, \quad h \in L^2(\mathbb{R})$$

and let $B : \mathbb{C}^N \to L^2(\mathbb{R})$ denote the bounded operator

(87.2) $$(Ba)(x) = \sum_{j=0}^{N-1} \phi_j(x)\, a_j \quad , \quad a = (a_0, \ldots, a_{N-1})^T \in \mathbb{C}^N$$

Then AB maps $\mathbb{C}^N \to \mathbb{C}^N$ and for $a \in \mathbb{C}^N$

(87.3) $$((AB)(a))_j = \left(A\left(\sum_{k=0}^{N-1} \phi_k(\cdot)\, a_k\right)\right)_j$$

$$= \sum_{k=0}^{N-1} a_k\, (A\phi_k)_j$$

$$= \sum_{k=0}^{N-1} a_k \left(\int \phi_j(x) g(x) \phi_k(x) dx \right)$$

and we see from (86.1) that

(88.1) $$\langle f \rangle = \det \left(1_{\mathbb{C}^N} + AB \right)$$

On the other hand $BA : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and for $h \in L^2(\mathbb{R})$

$$(BAh)(x) = \left(B \left(\left(\int \phi_j(y) g(y) h(y) dy \right)_{j=0}^{N-1} \right) \right)(x).$$

$$= \sum_{j=0}^{N-1} \phi_j(x) \int \phi_j(y) g(y) h(y) dy$$

$$= \int K(x,y) g(y) h(y) dy$$

where

(88.2) $$K(x,y) = \sum_{j=0}^{N-1} \phi_j(x) \phi_j(y)$$

is the so-called correlation kernel for the ensemble: $K(x,y)$ plays a central role in RMT.

Thus

(88.3) $$BA = K \chi_g$$

where χ_g denotes multiplication by $g(x)$ and K denotes the operator with kernel K i.e. $Kh(x) = \int K(x,y)h(y)\,dy$.

Assembling the above results we obtain the key formula

$$(89.1) \qquad \langle f \rangle = \det\left(1_{L^2(\mathbb{R})} + K\chi_g\right)$$

Note that as A (and also B!) is finite rank, A is trace class and the Fredholm determinant in (89.1) is well-defined. There are analogous, but more complicated, formulae for $\beta=1$ and $\beta=4$ (see Ref (3)).

If Ω is a Borel set in $\mathbb{R}$ and

$$g(x) = -\chi_\Omega$$

then we see from (89.1) that the gap probability considered before is given by

$$(89.2) \qquad \text{Prob}\,(\text{no eigenvalues in } \Omega)$$

$$= \left\langle \prod_{i=1}^{N} \left(1 - \chi_\Omega(\lambda_i)\right) \right\rangle, \quad \chi_\Omega = \text{characteristic function of } \Omega$$

$$= \det\left(1_{L^2(\mathbb{R})} - K\chi_\Omega\right)$$

Note that if $\Omega = \phi$, then RHS $= 1$ i.e.

Prob $\{$no eigs in $\phi\} = 1$, as it should be. On the other hand for $\Omega = \mathbb{R}$, Prob $\{$no eigs in $\mathbb{R}) = 0$. On the other hand for any $k = 0, 1, \ldots, N-1$,

$$K\phi_k(x) = \int \sum_0^{N-1} \phi_j(x)\,\phi_j(y)\,\phi_k(y)\,dy$$

$$= \sum_0^{N-1} \phi_j(x)\ \delta_{jk}$$

$$= \phi_k(x)$$

Thus 1 is an eigenvalue of K (of multiplicity $N > 0$) and so $\det(I - K) = 0$, as it should be.

Now take $\Omega = (a, \infty)$ for any $a \in \mathbb{R}$.

Then clearly

$$\text{Prob}\,(\text{no eigs in } (a, \infty)) = \text{Prob}\,(\lambda_{max} \le a)$$

where λ_{max} is the largest eigenvalue of M. Thus

(90.1) $$\text{Prob}\,(\lambda_{max} \le a) = \det\left(1_{L^2(\mathbb{R})} - K\chi_{(a,\infty)}\right)$$

This is a key formula in RMT for unitary ensembles.

Exercise

If $\Omega = \Omega_1 \cup \dots \cup \Omega_k$ is a union of disjoint sets in $\mathbb{R}$, $\Omega_j \cap \Omega_h = \phi$ for $j \neq h$, then we have computed

$$\text{Prob}\,(\, 0 \text{ eigs in } \Omega_1, \dots, 0 \text{ eigs in } \Omega_k)$$

$$= \text{Prob}\,\Big(\text{no eig's in } \Omega = \bigcup_{j=1}^{k} \Omega_j \Big).$$

Derive an explicit formula for

$$\text{Prob}\,\big(\, n_1 \text{ eigs in } \Omega_1, \dots, n_k \text{ eigs in } \Omega_k \big)$$

for any non-negative integers $n_1, \dots, n_k$. (see ref (3) pp 86-88)

We now show how to compute correlation functions for unitary ensembles (recall that 2-point correlation functions arose in the study of the non-trivial zeros of the Riemann zeta function in the first lecture).

If $P_N(x)\,dx$ is the probability distribution for a system of N identical random particles

$$P_N(x_1, \dots, x_N) \quad \text{symmetric in the } x_i\text{'s}$$

then the n-point correlation function

$$R_n = R_n(x_1, \dots, x_n) \quad , \quad 1 \le n \le N$$

for $P_N(x)\,dx$ is defined by

(92.1) $$R_n(x_1, \dots, x_n) = \frac{N!}{(N-n)!} \int \cdots \int P_N(x_1, \dots, x_n, x_{n+1}, \dots, x_N)\, dx_{n+1} \cdots dx_N$$

Note that $\int R_n(x_1, \dots, x_n)\, dx_1 \cdots dx_n \neq 1$ & hence R_n is <u>not</u> a prob. distrib (more later!).

Correlation functions are useful whenever we want to focus on $n < N$ particles and "average out"