the remaining particles: Indeed, suppose $F = F(x_1, \ldots, x_n)$ is a symmetric function of $x_1, \ldots, x_n$. Then

$$\frac{1}{n!} \int F(x_1, \ldots, x_n) R_n(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

$$= \frac{N!}{(N-n)!\, n!} \int F(x_1, \ldots, x_n)\, P_N(x_1, \ldots, x_n, x_{n+1}, \ldots, x_N)\, dx_1 \ldots dx_n dx_{n+1} \ldots dx_N$$

$$= \int \sum_{1 \le i_1 < \cdots < i_n \le N} F(x_{i_1}, \ldots, x_{i_n})\, P_N(x_1, \ldots, x_N)\, dx_1 \ldots dx_N.$$

Thus

(93.1) $$\mathbb{E}_{xp}\, \hat{F} = \frac{1}{n!} \int F(x_1, \ldots, x_n) R_n(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

where

(93.2) $$\hat{F}(x_1, \ldots, x_N) = \sum_{1 \le i_1 < \cdots < i_n \le N} F(x_{i_1}, \ldots, x_{i_n})$$

is the symmetric extension of $F(x_1, \ldots, x_n)$ to N variables.

Suppose $x_1^0 < x_2^0 < \cdots < x_n^0$ and let $\delta > 0$ be small. Let χ_j^0 be the characteristic function of the (disjoint) sets $(x_j^0 - \frac{\delta}{2}, x_j^0 + \frac{\delta}{2})$. Let

$$F(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \prod_{j=1}^{n} \chi_j^0 (x_{\sigma_j})$$

Clearly F is symmetric

Then $\cancel{\boxed{\text{from (93.2)}}}$ we have from (93.2)

$$\delta^n R_n(x_1^0, \ldots, x_n^0) \sim \frac{1}{n!} \int F(x_1, \ldots, x_n) R_n(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

$$= \int \hat{F}(x_1, \ldots, x_N) \, P_N(x_1, \ldots, x_N) dx_1 \ldots dx_N$$

$$= \int_{x_1 < \cdots < x_N} \hat{F}(x_1, \ldots, x_N) \, \hat{P}_N(x_1, \ldots, x_N) dx_1 \ldots dx_N$$

Insert Remark 94+

(where $\hat{P}_N = N! \, P_N$)

$$= \int_{x_1 < \cdots < x_N} \sum_{1 \le i_1 < \cdots < i_n \le N} F(x_{i_1}, \ldots, x_{i_n}) \, \hat{P}_N(x_1, \ldots, x_N) dx_1 \ldots dx_N$$

Now as $x_1 < \cdots < x_N$, $F(x_{i_1}, \ldots, x_{i_n}) = \prod_{j=1}^{n} \chi_j^0 (x_{i_j})$. Hence

$$\delta^n R_n(x_1^0, \ldots, x_n^0) = \sum_{x_1 < \cdots x_N} \left[\sum_{1 \le i_1 < \cdots < i_n \le N} \left(\prod_{j=1}^{n} \chi_j^0 (x_{i_j}) \right) \right] \hat{P}_N(x_1, \ldots, x_N) dx_1 \ldots dx_N$$

$$= \sum_{x_1 < \cdots < x_N} \left[\chi_1^0(x_1) \cdots \chi_n^0(x_n) + \cdots + \chi_1^0(x_{N-n+1}) \cdots \chi_1^0(x_N) \right]$$

$$\hat{P}_N(x_1, \ldots, x_N) \, dx_1 \ldots dx_N$$

$\simeq$ Prob (exactly 1 eigenvalue in each of the intervals $(x_j^0 - \delta/2, x_j^0 + \delta/2)$)

why?

<u>Note:</u> Insert on p. 94

In the case of RMT $P_N(\lambda_1, \ldots, \lambda_N)\, d\lambda_1 \cdots d\lambda_N$ only has <u>physical</u> meaning, even though $P_N(\lambda_1, \ldots, \lambda_N)$ is symmetric in the λ_j's, only when $\lambda_1 < \cdots < \lambda_N$. Indeed, remember that the map $M \mapsto (\Lambda(M), O(M))$ always specifies the eigenvalues in some order, in particular, $\lambda_1(M) < \cdots < \lambda_N(M)$.

When we compute the expectation $\mathbb{E}\mathrm{xp}\, f$ for some quantity $f(\lambda_1, \ldots, \lambda_N)$ which is symmetric in $\lambda_1, \ldots, \lambda_N$, we have

(94+.1) $$\mathbb{E}\mathrm{xp}\, f = \int \cdots \int_{\lambda_1 < \cdots < \lambda_N} f(\lambda_1, \ldots, \lambda_N)\, P_N(\lambda_1, \ldots, \lambda_N)\, d\lambda_1 \cdots d\lambda_N$$

However as a <u>computational convenience</u> we observe that

(94+.2) $$\mathbb{E}\mathrm{xp}\, f = \frac{1}{N!} \int \cdots \int_{\mathbb{R}^N} f(\lambda_1, \ldots, \lambda_N)\, P_N(\lambda_1, \ldots, \lambda_N)\, d\lambda_1 \cdots d\lambda_N$$

Although (94+.2) is easier to manipulate, when we want to understand the <u>meaning</u> of the statistic $\mathbb{E}\mathrm{xp}\, f$, we must refer to (94+.1)

Thus

(95.1) $R_n(x_1^0, \ldots, x_n^0)$ is the density of the probability that is one eigenvalue at each of the points

$$x_1^0, \ldots, x_n^0, \quad x_1^0 < \ldots < x_n^0$$

Note the following:

If $F = F(x_1) = \chi_\Omega(x_1)$, the characteristic function of $\Omega \subset \mathbb{R}$

$$\hat{F}(x_1, \ldots, x_N) = \sum_{i=1}^N F(x_i) = \sum_{i=1}^N \chi_\Omega(x_i) = \#\{i : x_i \in \Omega\}$$

Thus by (93.1)

(95.1) $$\mathbb{E}xp\,(\#\{i : x_i \in \Omega\}) = \int_\Omega R_1(x)\,dx$$

Bearing (94+) in mind, we also have for random matrix ensembles,

(95.2) $$\mathbb{E}xp\,(\#\{\lambda_i \in \Omega\}) = \int_\Omega R_1(x)\,dx$$

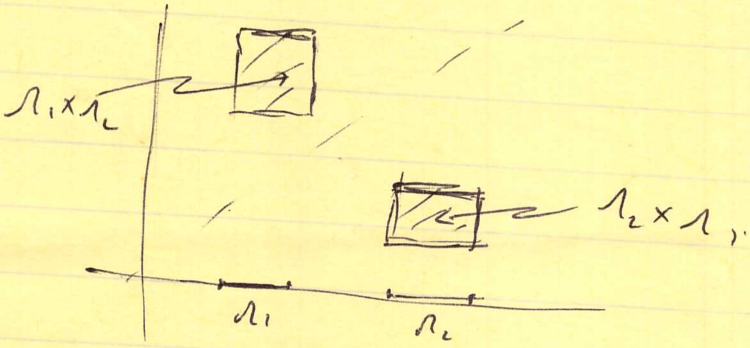
Also if Ω_1, Ω_2 are two disjoint sets in $\mathbb{R}$ and

$$F(x_1, x_2) = \chi_{\Omega_1}(x_1)\,\chi_{\Omega_2}(x_2) + \chi_{\Omega_1}(x_2)\,\chi_{\Omega_2}(x_1).$$

then

$$\hat{F}(x_1, \ldots, x_N) = \sum_{1 \le i_1 < i_2 \le N} \left[\chi_{\Omega_1}(x_{i_1})\,\chi_{\Omega_2}(x_{i_2}) + \chi_{\Omega_1}(x_{i_2})\,\chi_{\Omega_2}(x_{i_1})\right]$$

$= \#\{(i_1, i_2) : i_1 < i_2,\ (x_{i_1}, x_{i_2}) \in \Lambda_1 \times \Lambda_2 \cup \Lambda_2 \times \Lambda_1.\}$



Thus

(96.1) $$\mathbb{E}\text{xp}\ \big(\#\{\text{pairs } (i_1, i_2),\ i_1 < i_2 : \text{ either } x_{i_1} \in \Lambda_1 \text{ and } x_{i_2} \in \Lambda_2 \text{ or } x_{i_2} \in \Lambda_1 \text{ and } x_{i_1} \in \Lambda_2 \} \big)$$

$$= \frac{1}{2!} \int \left[\chi_{\Lambda_1}(x_1)\,\chi_{\Lambda_2}(x_2) + \chi_{\Lambda_1}(x_2)\,\chi_{\Lambda_2}(x_1) \right] R(x_1, x_2)\, dx_1\, dx_2 .$$

$$= \int_{\Lambda_1 \times \Lambda_2} R(x_1, x_2)\, dx_1\, dx_2$$

Insert 96+ $\rightarrow$

<u>Exercise</u>: Show how (96.1) changes if $\Lambda_1 = \Lambda_2$.

We now show how to compute $R_n(x_1, \dots, x_n)$ using

(89.1)
$$\langle f \rangle = \det\left(1_{L^2(\mathbb{R})} + K\chi_g \right)$$

where $f(M) = \det(I + g(M))$ and K is given in (88.2)

Insert on p96

Again bearing (94+) in mind, we have for random matrix ensemble,

$$\int_{x_1 < \cdots < x_N} \hat{F}(x_1, \ldots, x_N)\, \hat{P}_N(x)\, d^N x .$$

$$= \int_{\mathbb{R}^N} \hat{F}(x_1, \ldots, x_N)\, P_N(x)\, d^N x , \qquad P_N = \frac{1}{N!} \hat{P}_N$$

(96.1) $$= \int_{\Lambda_1 \times \Lambda_2} R(x_1, x_2)\, dx_1\, dx_2$$

Now suppose for definiteness that Λ_1 lies to the left of Λ_2

Λ_1 Λ_2

Then for $x_1 < \cdots < x_N$

$$\hat{F}(x_1, \ldots x_N) = \sum_{1 \le i_1 < i_2 \le N} \chi_{\Lambda_1}(x_{i_1})\, \chi_{\Lambda_2}(x_{i_2})$$

$= \#\{$ ordered pairs of eigenvalues, (x_{i_1}, x_{i_2}), $x_{i_1} < x_{i_2}$ such that $(x_{i_1}, x_{i_2}) \in \Lambda_1 \times \Lambda_2 \}$

Hence

(96+.1) $\mathbb{E}\mathrm{xp}\, \{ \#\{$ ordered pair of eigenvalues, (x_{i_1}, x_{i_2}), $x_{i_1} < x_{i_2}$, such that $(x_{i_1}, x_{i_2}) \in \Lambda_1 \times \Lambda_2 \}$.

$= \int_{\Lambda_1 \times \Lambda_2} R(x_1, x_2)\, dx_1\, dx_2$.

More explicitly for any $g \in L^\infty(\mathbb{R})$.

(97.0) $$\int \prod_{i=1}^{N} (1+g(x_i)) \, P_N(x_1, \ldots, x_N) \, d^N x = \det(1 + K \chi_g)_{L^2(\mathbb{R})}$$

where

$$P_N(x) d^N x = \frac{\left(\prod_{i=1}^{N} w(x_i)\right) |V(x)|^2 d^N x}{\int \left(\prod_{i=1}^{N} w(y_i)\right) |V(y)|^2 d^N y}$$

Choose g such that

$$1+g = \gamma_0 \chi_0 + \cdots + \gamma_k \chi_k \quad \text{for some } k$$

where χ_i are the characteristic functions of disjoint (Borel) sets Ω_i, $0 \le i \le k$, in $\mathbb{R}$, such that $\mathbb{R} = \bigcup_{i=0}^{k} \Omega_i$.
Here $\gamma_i \in \mathbb{R}$, $i = 0, \ldots, k$.
Clearly

(97.1) $$g = (\gamma_0 - 1)\chi_0 + \cdots + (\gamma_k - 1)\chi_k$$

$$= \sum_{i=0}^{k} \eta_i \chi_i, \qquad \eta_i = \gamma_i - 1, \quad 0 \le i \le k.$$

For any $1 \le j \le N$, let

(97.2) $$\sigma_j(\xi_1, \ldots, \xi_N) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le N} \xi_{i_1} \cdots \xi_{i_j}$$

denote the jth elementary symmetric function and set $\sigma_0 = 1$.

We have

(98.1) $$\prod_{i=1}^{N} (1+\xi_i) = \sum_{j=0}^{N} \sigma_j(\xi_1, \dots, \xi_N) .$$

Thus

$$\int \prod_{i=1}^{N} (1+g(x_i)) \, P_N(x) \, d^N x$$

$$= \sum_{j=0}^{N} \sum_{1 \le i_1 < \dots < i_j \le N} \int g(x_{i_1}) \dots g(x_{i_j}) \, P_N(x) \, d^N x$$

$$= \sum_{j=0}^{N} \binom{N}{j} \int g(x_1) \cdots g(x_j) \, P_N(x) \, d^N x$$

(by symmetry)

$$= \sum_{j=0}^{N} \binom{N}{j} \frac{(N-j)!}{N!} \int g(x_1) \cdots g(x_j) \, R_j(x_1, \dots, x_j) \, dx_1 \cdots dx_j$$

→ Insert 98+, 98++

Substituting (98++.2) for $\prod_{i=1}^{j} g(x_i)$ we find (exercise: see ref(3) p87)

(98.2) $$\int \prod_{i=1}^{N} (1+g(x_i)) \, P_N(x) \, d^N x$$

$$= \sum_{\substack{n_0, n_1, \dots, n_k \ge 0 \\ 0 \le |n| \le N}} \frac{\eta_0^{n_0} \eta_1^{n_1} \cdots \eta_k^{n_k}}{|n|!} \int_{\mathbb{R}^{|n|}} R_{|n|}(x_1, \dots, x_{|n|}) \times \chi_{\{n_0, n_1, \dots, n_k \text{ of } \{x_1, \dots, x_{|n|}\} \text{ lie in } \Delta_0, \Delta_1 \dots \Delta_k \text{ resp}\}} \, d^{|n|} x$$

Insert on p 98

Now

$$(98+.1) \qquad \prod_{i=1}^{j} g(x_i) = \prod_{i=1}^{j} \left(\eta_0 \chi_0(x_i) + \cdots + \eta_k \chi_k(x_i)\right)$$

$$= \sum_{i_1,\dots,i_j=0}^{k} \eta_{i_1}\cdots\eta_{i_j}\, \chi_{i_1}(x_1)\cdots\chi_{i_j}(x_j)$$

$$= \sum_{\substack{n_0,n_1,\dots,n_k\geq 0\\ \sum_{i=0}^{k} n_i = j}} \; \sum_{\substack{0\leq i_1,\dots,i_j\leq k\\ \#\{q:\, i_q=0\}=n_0\\ \vdots \\ \#\{q:\, i_q=k\}=n_k}} \eta_{i_1}\cdots\eta_{i_j}\, \chi_{i_1}(x_1)\cdots\chi_{i_j}(x_j)$$

$$= \sum_{\substack{n_0,n_1,\dots,n_k\geq 0\\ \sum_{i=0}^{k} n_i = j}} \eta_0^{n_0}\eta_1^{n_1}\cdots\eta_k^{n_k}\; E(n_0,\dots,n_k;x)$$

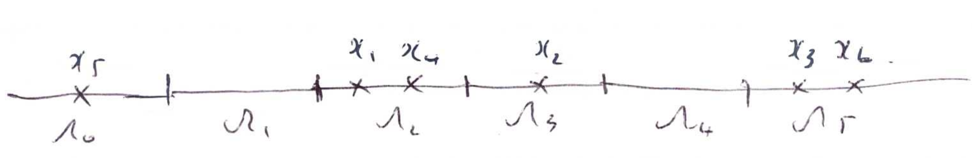
where

$$(98+.2) \qquad E(n_0,n_1,\dots,n_k;x) = \sum_{\substack{0\leq i_1,\dots,i_j\leq k\\ \#\{q:\, i_q=0\}=n_0\\ \vdots \\ \#\{q:\, i_q=k\}=n_k}} \chi_{i_1}(x_1)\cdots\chi_{i_j}(x_j)$$

Consider, for example, the case where $k=5$ and $j=6$ and

$n_0=1 \quad n_1=0 \quad n_2=2 \quad n_3=1 \quad n_4=0 \quad n_5=2$, $\sum_{q=0}^{k} n_q = j = 6$

and with $\{x_1,\dots,x_6\}$ arranged as follows



Now clearly $\chi_{i_1}(x_1)\,\chi_{i_2}(x_2)\cdots\chi_{i_j}(x_6) = 1$ if and only

if $\quad i_1=2 \quad i_2=3 \quad i_3=5 \quad i_4=2 \quad i_5=0 \quad i_6=5$

In particular only <u>one</u> term in (98+2.) contributes. We conclude that

$$(98++.1) \quad E(n_0, n_1, \dots, n_k : x) = \chi_{\{x=\{x_1,\dots,x_j\} :\ n_q \text{ of the } x_i\text{'s are in } \Omega_q,\ 0 \le q \le k\}}(x)$$

Thus

$$(98++.2) \quad \prod_{i=1}^{j} g(x_i) = \sum_{\substack{n_0, n_1, \dots, n_k \ge 0 \\ \sum_{i=0}^{k} n_i = k}} \eta_0^{n_0} \eta_1^{n_1} \cdots \eta_k^{n_k} \, \chi_{\{x=(x_1,\dots,x_j) :\ n_q \text{ of the } x_i\text{'s} \in \Omega_q,\ 0 \le q \le k\}}(x)$$

where $|n| = n_0 + n_1 + \cdots + n_k$

On the other hand, by the Fredholm expansion of a determinant,

$$\det(1+Kg) = \sum_{j=0}^{\infty} \frac{1}{j!} \int_{\mathbb{R}^j} \det \begin{bmatrix} K(x_1,x_1) & \cdots & K(x_1,x_j) \\ K(x_j,x_1) & \cdots & K(x_j,x_j) \end{bmatrix} \prod_{i=1}^{j} g(x_i)\, d^j x$$

$$= \sum_{j=0}^{N} \frac{1}{j!} \int_{\mathbb{R}^j} \det\left(K(x_i,x_k)\right)_{i,k=1}^{j} \prod_{1}^{j} g(x_i)\, d^j x$$

Here we have used the fact that $\det\left(K(x_i,x_j)\right)_{i,k=1}^{j} = 0$ if $j > N$ (why?).

Again expanding out $g(x)$ using (98+1.2) we find as above

(99.1) $$\det(1+K\chi_g) = \sum_{\substack{n_0,n_1,\ldots,n_k \geq 0 \\ 0 \leq |n| \leq N}} \frac{\eta_0^{n_0} \cdots \eta_k^{n_k}}{|n|!} \int_{\mathbb{R}^{|n|}} \det\left(K(x_i,x_k)\right)_{i,k=1}^{|n|}$$

$$\times \chi_{\{n_0,n_1,\ldots,n_k \text{ of } \{x_1,\ldots,x_{|n|}\} \text{ lie in } \Omega_0,\Omega_1,\ldots,\Omega_k \text{ resp}\}}\, d^{|n|}x$$

Equating (98.2) and (99.1), and comparing coefficients, we find in particular for $k \leq N$, $n_0 = 0$, $n_1 = \cdots = n_k = 1$

(100.1) $$0 = \int_{\mathbb{R}^k} \Delta_k(x_1,\dots,x_k)\, \chi_{\{\Omega_1,\dots,\Omega_k \text{ have } \underline{\text{one}} \text{ of } \{x_1,\dots,x_k\},\ \Omega_0 \text{ has none}\}}\, d^k x$$

where $$\Delta_k(x_1,\dots,x_k) = R_k(x_1,\dots,x_k) - \det\left(K(x_i,x_j)\right)_{i,j=1}^{k}$$

As $\Delta_k(x_1,\dots,x_k)$ is symmetric, (100.1) $\Rightarrow$

(100.2) $$\int_{\substack{x_1\le\dots\le x_k \\ \#\{i:\, x_i\in\Omega_j\}=1 \\ 1\le j\le k}} \Delta_k(x_1,\dots,x_k)\, d^k x = 0$$

Let $\Omega_j = (a_j, b_j)$, $1\le j\le k$, be disjoint intervals ordered from the left i.e.

$$a_1<b_1<a_2<b_2<\dots<a_k<b_k.$$

and inserting these Ω_j's into (100.2) and letting $b_j\downarrow a_j$ we obtain

$$\Delta_k(a_1,\dots,a_k)=0$$

for all $a_1<\dots<a_k$ and hence for all $a_1,\dots,a_k$ by symmetry. We conclude that for $k\ge 1$,

(100.3) $$R_k(x_1,\dots,x_k) = \det\left(K(x_i,x_j)\right)_{1\le i,j\le k}$$

Exercise Use the above calculations to rederive (95.1)

Remark! For other proofs of this result see ref 3 pp 96→98 and also ref 2 pp103–108 (this calculation is taken from [Meh]).

Insert 101+ 101++, 101+++

The above calculations show that in order to evaluate key eigenvalue statistics for Unitary Ensembles we must understand the asymptotic behavior of the correlation kernel

$$K(x,y) = \sum_{j=0}^{N-1} \phi_j(x)\, \phi_j(y)$$

where $\phi_j(x) = p_j(x)\,(w(x))^{\frac{1}{2}}$, and the p_j's are orthonormal wrt the weight $w(x)$,

$$\int_{\mathbb{R}} p_j(x)\, p_k(x)\, w(x)\, dx = \delta_{ij} \quad , \quad 0 \le i, j < \infty .$$

Thus the problem of the asymptotics of eigenstatistics reduces, for Unitary Ensembles to the classical problem of the asymptotics of orthogonal polynomials (OP's)

We now compute

Prob $\{ n_1$ eigenvalues in $\Omega_1, \ldots, n_k$ eigenvalues in $\Omega_k \}$

where again the Ω_i's are disjoint and $\sum_{j=1}^{k} n_j = N$

Set $n_0 = N - \sum_{j=1}^{k} n_j$ and set $\Omega_0 = \mathbb{R} \setminus \bigcup_{j=1}^{k} \Omega_j$

Again letting χ_i be the characteristic function of Ω_j, $0 \le j \le k$, we have, using (98++.1)

Prob $\{ n_1$ eig's in $\Omega_1, \ldots, n_k$ eig's in $\Omega_k \}$

$$= \frac{\int_{x_1 \le x_2 \le \cdots \le x_N} E(n_0, n_1, \ldots, n_k : x) \prod_{i=1}^{N} w(x_i) \, |V(x)|^2 \, d^N x}{\int_{x_1 \le \cdots \le x_N} \prod_{i=1}^{N} w(x_i) \, |V(x)|^2 \, d^N x}$$

$$= \frac{\int_{\mathbb{R}^N} E(n_0, \ldots, n_k : x) \prod_{i=1}^{N} w(x_i) \, |V(x)|^2 \, d^N x}{\int_{\mathbb{R}^N} \prod_{i=1}^{N} w(x_i) \, |V(x)|^2 \, d^N x}$$

Now for

$$F(x, \gamma_0, \ldots, \gamma_k) = \prod_{j=1}^{N} (\gamma_0 \chi_0 + \cdots + \gamma_k \chi_k)(x_j)$$

we have for $\sum_{i=1}^{k} n_i \le N$

$$E(n_0, \ldots, n_k : x) = \frac{1}{n_1! \cdots n_k!} \left. \frac{\partial^{n_1 + \cdots + n_k}}{\partial \gamma_1^{n_1} \cdots \partial \gamma_k^{n_k}} \right|_{\substack{\gamma_0 = 1 \\ \gamma_1 = \cdots = \gamma_k = 0}} F(x : \gamma_0, \ldots, \gamma_k)$$

where we have used (98+.1) with $j = N$

$$F = \prod_{i=1}^{N} (\gamma_0 \chi_0 + \cdots + \gamma_k \chi_k)(x_i)$$

$$= \sum_{\substack{n_0, n_1, \ldots, n_k \geq 0 \\ \sum_{i=1}^{k} n_i = N}} \gamma_0^{n_0} \cdots \gamma_k^{n_k} \, E(n_0, \ldots, n_k : x).$$

Thus

Prob { n_1 eigs in $\Delta_1, \ldots, n_k$ eigs in Δ_k }

$$= \frac{1}{n_1! \cdots n_k!} \left. \frac{\partial^{n_1 + \cdots + n_k}}{\partial \gamma_1^{n_1} \cdots \partial \gamma_k^{n_k}} \right|_{\substack{\gamma_0 = 1 \\ \gamma_1 = \cdots = \gamma_k = 0}} \frac{\int_{\mathbb{R}^N} \prod_{i=1}^{N} (\gamma_0 \chi_0 + \cdots + \gamma_N \chi_N)(x_i) \prod_{i=1}^{N} w(x_i) |V(x)|^2 d^N x}{\int_{\mathbb{R}^N} \prod_{i=1}^{N} w(x_i) \, |V(x)|^2 \, d^N x}$$

Setting $1 + g = \gamma_0 \chi_0 + \cdots + \gamma_k \chi_k$
and hence at $\gamma_0 = 1$,

$$g = \eta_1 \chi_1 + \cdots + \eta_k \chi_k \qquad \text{where} \quad \eta_j = \gamma_j - 1, \quad 1 \leq j \leq k$$

It follows now from (89.1)

$$\langle f \rangle = \langle 1 + g \rangle = \det \left(I + K \chi_g \right)$$

where K is the correlation kernel in (88.2)

$$K(x, y) = \sum_{j=0}^{N-1} \phi_j(x) \phi_j(y)$$

Thus, finally,

(101+++.1) Prob $\{n_1$ eigs in $\Delta_1, \ldots, n_k$ eigs in $\Delta_k\}$

$$= \frac{1}{n_1! \cdots n_k!} \frac{\partial^{n_1+\cdots+n_k}}{\partial \eta_1^{n_1} \cdots \partial \eta_k^{n_k}} \bigg|_{\eta_1 = \cdots = \eta_k = -1} \det\left(1 + K \sum_{i=1}^{k} \eta_i \chi_i\right)$$

(Ref: Szegö, "Orthogonal Polynomials").

For the next couple of lectures we will consider this problem. A key object that controls the asymptotics of OP's is the so-called equilibrium measure (see Ref 2, Chap 6 : see also Saff & Totik "Logarithmic potentials and external fields" for the general theory)

We will see eventually that this quantity is intimately related to the density of states for RMT and also to the one-point correlation function $R_1(x)$. The calculations below are taken from ref(2), which in turn are based on work of K. Johansson (see ref(2)).

In the calculations that follow we will always assume that the probability density $P_N(M)\,dM$ varies with N in the following way

$$P_N(M)\,dM = \frac{1}{Z_N} e^{-N \operatorname{tr} V(M)}\,dM \tag{102.1}$$

As all our calculations so far have assumed that N is given and fixed, they all remain valid: we must just set

$$w = w_N = e^{-N \operatorname{tr} V(M)}$$

After integrating out the eigenvectors we obtain as before a probability measure on the eigenvalues

$$\hat{P}_N(\lambda)\, d^N\lambda = \frac{1}{\hat{Z}_N} e^{-N \sum V(\lambda_i)} \prod_{i<j} (\lambda_i - \lambda_j)^2 \, d^N\lambda$$

where (after symmetrizing)

(103.1) $$\hat{Z}_N = \int_{\mathbb{R}^N} e^{-N \sum V(\lambda_i)} \prod_{i<j} |\lambda_i - \lambda_j|^2 \, d^N\lambda$$

$$= \int_{\mathbb{R}^N} e^{-N^2 H(\lambda)} \, d^N\lambda$$

where $$H(\lambda) = \frac{1}{N^2} \sum_{i \neq j}^{N} \log |\lambda_i - \lambda_j|^{-1} + \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i)$$

Let

(103.2) $$\mu_\lambda = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$$

be the <u>normalized counting measure</u> for the eigenvalues. and note that $H(\lambda)$ can be expressed as follows:

(104.1) $$H(\mu_\lambda) = \iint_{t\neq s} \log|t-s|^{-1} d\mu_\lambda(t)\, d\mu_\lambda(s) + \int V(t)\, d\mu_\lambda(t)$$

Note that the scaling in the potential

$$e^{-\operatorname{tr} V} \to e^{-N \operatorname{tr} V}$$

is chosen so that the terms in (104.1) are balanced.

Intuitively the leading contribution to the partition function (103.1) as $N \to \infty$ comes from those λ_i's for which $H(\mu_\lambda)$ is a minimum. Thus we are led to consider the following energy minimization problem

(104.2) $$E^V = \inf_{\mu \in M_1(\mathbb{R})} H(\mu)$$

where $H(\mu) = \iint \log|t-s|^{-1} d\mu(t)\, d\mu(s) + \int V(t)\, d\mu(t)$

and $M_1(\mathbb{R}) = \{ \mu \text{ is a Borel measure on } \mathbb{R} : \int d\mu = 1 \}$

We will show eventually that (103.2) has a unique minimizer $\mu = \mu^{eq}$, the <u>equilibrium measure</u> mentioned above. From the definition of $H(\mu)$, μ^{eq} has an electrostatic interpretation: it is the equilibrium configuration for electrons with logarithmic electrostatic repulsion

$$\iint \log |t-s|^{-1} \, d\mu^{eq}(t) \, d\mu^{eq}(s)$$

in an external field $\int V(t) \, d\mu^{eq}(t)$. As already indicated $d\mu^{eq}$ is intimately related to a variety of problems in RMT, and also in analysis. The existence and uniqueness of the solution of the variational problem (104.2) relies ultimately on the fact that we are dealing with a (constrained) <u>convex</u> minimization problem.