(see ref(21))

To begin, let $\mathcal{M}(\mathbb{R})$ be the set of finite Borel measures on $\mathbb{R}$. By Helly's selection theorem (exercise!) every bounded sequence of measures $\{\mu_n\}$ on $\mathbb{R}$

$$\int d\mu_n \le c < \infty$$

has a vaguely convergent subsequence i.e. a subsequence $\{\mu_{n(k)}\}$ and a measure μ, $\int \mu \le c$, such that

$$\int g(s)\, d\mu_{n,k}(s) \to \int g(s)\, d\mu(s)$$

for all $g \in C_c$, the set of all continuous (& hence bounded) function on $\mathbb{R}$ with compact support. One writes

(106.1) $$\mu_{n(k)} \xrightarrow{v} \mu$$

and says that balls of fixed size in $\mathcal{M}(\mathbb{R})$ are vaguely sequentially compact.

Exercise: Prove the Helly selection theorem

(i) directly and (ii) by using the Banach - Alaoglu Th^m.

Note that under vague convergence, a sequence of prob. measures $\{\mu_n\}$, $\int d\mu_n = 1$, can lose mass.

We always have $\forall g \in C_c$, $|\int g\, d\mu| \leq \|g\|_\infty$ if $\mu_n \xrightarrow{v} \nu$ (prob. measures)

and so $\int d\mu \leq 1$. But $\int d\mu < 1$ is a possibility:

eg if $\mu_n = \delta_n$: then $\mu_n \xrightarrow{v} \mu = 0$ & so $\int d\mu = 0$!

We will need a criterion that mass is not lost.

We say that a sequence $\{\mu_n\}$ of prob. measures on $\mathbb{R}$ is tight if for all $\varepsilon > 0$, $\exists$ $M = M_\varepsilon$ st

(107.1) $$\overline{\lim_n} \int_{|x| \geq M} d\mu_n \leq \varepsilon$$

Note that $\{\mu_n\} = \{\delta_n\}$ is not tight.

Th^m 107.2 Let $\{\mu_n\}$ be a sequence of prob. measures on $\mathbb{R}$. Then $\{\mu_n\}$ is tight $\Leftrightarrow$ every vaguely convergent subsequence of $\{\mu_n\}$ converges to a probability measure.

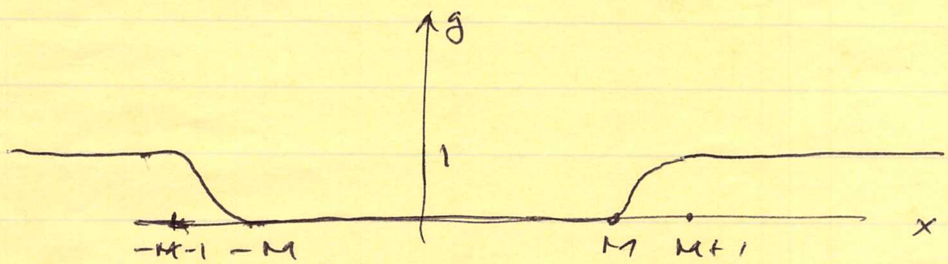
Proof: Suppose $\{\mu_n\}$ is tight and that $\mu_{n(k)} \xrightarrow{v} \mu$

Let $\varepsilon > 0$ be given and choose M to satisfy (107.1).

Let $g(x) \in C_b = \{$bded continuous functions on $\mathbb{R}\}$

have the following properties:

- $g(x) = 1$ for $|x| \geq M+1$
- $0 \leq g(x) \leq 1$ $\forall x$
- $g(x) = 0$ for $|x| \leq M$



Then clearly

$$\overline{\lim_n} \int g(s)\, d\mu_n \leq \varepsilon.$$

which implies

$$\underline{\lim_n} \int (1 - g(s))\, d\mu_n(s) \geq 1 - \varepsilon.$$

But $1-g \in C_c$ and hence

$$1 \geq \int d\mu(x) \geq \int (1-g(s))\,d\mu(s) = \lim_{k\to\infty} \int (1-g(s))\,d\mu_{n(k)}(s) \geq 1-\varepsilon .$$

As $\varepsilon > 0$ is arbitrary, we conclude that $\int d\mu = 1$ ie μ is a probability measure

Conversely, suppose that every vaguely convergent subsequence of μ_n converges to a probability measure and that $\{\mu_n\}$ is not tight. Then $\exists\ \varepsilon_0$, $0<\varepsilon_0<1$, and a subsequence $\{\mu_{n(k)}\}$ such that $\int_{|x|>k} d\mu_{n(k)} > \varepsilon_0$. By Helly's theorem, we can assume that $\mu_{n(k)} \xrightarrow{v} \mu \in M(\mathbb{R})$. Suppose $g \in C_c$. Then for k suff. large, $\operatorname{supp} g \subset [-k,k]$, and

$$\left| \int g(x)\,d\mu_{n(k)}(x) \right| = \int_{|x|\leq k} g(x)\,d\mu_{n(k)}(x)$$

$$\leq \|g\|_\infty \left(1 - \int_{|x|>k} d\mu_{n(k)}(x)\right)$$

$$\leq \|g\|_\infty (1-\varepsilon_0)$$

Letting $k\to\infty$, we see that $\left|\int g(x)\,d\mu(x)\right| \leq \|g\|_\infty (1-\varepsilon_0)$

ie $\int d\mu < 1-\varepsilon_0 < 1$, which is a contradiction. $\square$ (110)

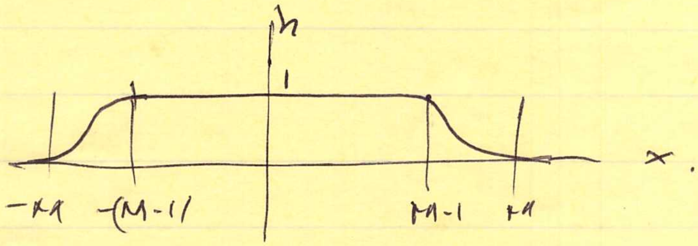
Note that if a sequence of prob. measures $\{\mu_n\}$ converges vaguely to a probability measure μ, then in fact μ_n converges weakly μ ie

$$\int g\, d\mu_n \to \int g\, d\mu$$

(Symbolically $\mu_n \to \mu$.)

for all $g \in C_b$. Indeed, fix $g \in C_b$ and let $\varepsilon > 0$ be given. Choose $M > 1$ so that $\|g\|_\infty \int_{|x| \geq M-1} d\mu(x) < \varepsilon$.

Let $h \in C_c$, $0 \leq h(x) \leq 1$, $h(x) = 1$ for $|x| \leq M-1$ and $h(x) = 0$ for $|x| > M$



Then

$$\int (1-h(x))\, d\mu_n(x) = 1 - \int h(x)\, d\mu_n(x)$$

$$\to 1 - \int h(x)\, d\mu(x) = \int (1-h(x))\, d\mu(x)$$

as $\int d\mu = \int d\mu_n = 1$ and $\mu_n \xrightarrow{v} \mu$. Now

$$\|g\|_\infty \overline{\lim_n} \int (1-h(x))\, d\mu_n(x)$$

$$= \|g\|_\infty \int (1-h(x))\, d\mu(x)$$

$$\leq \|g\|_\infty \int_{|x|\geq M-1} d\mu(x) < \varepsilon .$$

and hence

$$\int g(x)\, d\mu_n(x) = \int g(x) h(x)\, d\mu_n(x) + a_n \quad \text{(where } \overline{\lim_n} |a_n| < \varepsilon \text{)}$$

$$= \int g(x) h(x)\, d\mu(x) + o(1) + a_n$$

$$= \int g(x)\, d\mu(x) + O(\varepsilon) + o(1) + a_n$$

Thus

$$\overline{\lim_n} \left| \int g(x)\, d\mu_n(x) - \int g(x)\, d\mu(x) \right| \leq 2\varepsilon .$$

As $\varepsilon > 0$ is arbitrary, we conclude that $\mu_n \to \mu$.

It follows that we have proved the following basic result

Th^m III.1 Suppose the prob. measures $\{\mu_n\}$ are tight. Then $\{\mu_n\}$ has a subsequence that converges weakly to a probability measure.

We will now prove the existence of μ^{eq}.

We will assume that $V(x)$ is continuous and

(112.1) $$\frac{V(x)}{\log(x^2+1)} \to +\infty \quad \text{as } |x| \to \infty.$$

This is true in particular if

(112.2) $$V(x) = t_{2m} x^{2m} + t_{2m-1} x^{2m-1} + \cdots + t_0, \quad t_{2m} > 0$$

Note that condition (112.1) ensures that the measure

$$e^{-V(x)} dx$$

has finite moments (why?) and hence the measure

$e^{-\mathrm{tr}\, V(M)} dM$ has finite moments (why?).

Set

(112.3) $$\varphi^V(x) \equiv V(x) - \log(x^2+1)$$

It follows from (112.1) that

(112.4) $$\lim_{|x| \to \infty} \varphi^V(x) = +\infty \quad \text{and} \quad \varphi^V(x) \geq c_V > -\infty.$$

(as $V(x)$ is continuous)

Consider for $\mu \in M_1(\mathbb{R})$,

$$H(\mu) = \iint \log|t-s|^{-1} d\mu(t)\, d\mu(s) + \int V(s)\, d\mu(s)$$

$$= \iint k(t,s)\, d\mu(t)\, d\mu(s)$$

where

(113.1) $$k(t,s) = \log|t-s|^{-1} + \tfrac{1}{2} V(t) + \tfrac{1}{2} V(s)$$

As we will show below that

(113.2) $$k(t,s) \geq c_V$$

so that $\mu \mapsto H(\mu)$ is a well-defined map from $M_1(\mathbb{R})$ to $(-\infty, \infty]$

Theorem 113.3

Let $V(x)$ be a continuous function satisfying (112.1)

Then $\exists$ a unique prob. meas. $\mu = \mu^{ed}$ such that

(113.4) $$E^V \equiv \inf_{\mu \in M_1(\mathbb{R})} H(\mu) = H(\mu^{ed})$$

Moreover μ^{ed} has compact support.

Note: W/o generality, we can assume $c_V \geq 0$: this just amounts to changing E^V by a constant.

Proof:
$$(t-s)^2 = t^2 + s^2 - 2st$$
$$\leq t^2 + s^2 + 1 + s^2 t^2$$
$$= (1+t^2)(1+s^2)$$

and so

(114.1) $$|t-s| \leq \sqrt{1+t^2}\,\sqrt{1+s^2} \qquad \forall t,s \in \mathbb{R}$$

Thus
$$\log|t-s|^{-1} \geq -\tfrac{1}{2}\log(1+t^2) - \tfrac{1}{2}\log(1+s^2)$$

and hence

(114.2) $$k(t,s) \geq \tfrac{1}{2}\psi(s) + \tfrac{1}{2}\psi(t) \geq c_V$$

Thus $H(\mu) \geq c_V \quad \forall \mu \in \mathcal{M}_1(\mathbb{R})$ & so $E^V \geq c_V \geq 0$

Also, clearly, $E^V < \infty$. Indeed let $\mu_1(t) = \chi_{[0,1]}\,dt$

Then $\mu_1 \in \mathcal{M}_1(\mathbb{R})$ and

$$H(\mu_1) = \int_0^1\!\!\int_0^1 \log|t-s|^{-1}\,ds\,dt + \int_0^1 V(s)\,ds$$
$$= \int_0^1 ds \int_{-s}^{1-s} \log|u|^{-1}\,du + \int_0^1 V(s)\,ds$$
$$\leq \int_0^1 ds \int_{-1}^{1} \log|u|^{-1}\,du + \int_0^1 V(s)\,ds$$
$$< \infty$$

Thus $0 \leq c_V \leq E^V \leq H(\mu_1) < \infty$.

We first show that if $\mu_n \to \mu$, $\mu \in M_1(\mathbb{R})$, then

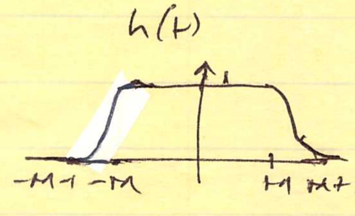
(115.0) $$\varliminf_n H(\mu_n) \geq H(\mu)$$

ie $\mu \mapsto H(\mu)$ is (weakly) <u>lower semi-continuous</u>.

For any L, we have

(115.1) $$H(\mu_n) \geq \iint \min\left(L, k(t,s)\right) d\mu_n(t)\, d\mu_n(s)$$

Now $g(t,s) \equiv \min\left(L, k(t,s)\right)$ is clearly a bounded, continuous function on $\mathbb{R}\times\mathbb{R}$. Let $\varepsilon>0$ be given, and chose M st $\displaystyle\int_{|x|\geq M} d\mu < \varepsilon$. Let $h(t) \in C_c(\mathbb{R})$ be such that



(115.2)
- $0 \leq h(t) \leq 1 \quad \forall t$
- $h(t) = 1 \quad$ for $|t| \leq M$
- $h(t) = 0 \quad$ for $|t| \geq M+1$

Now

(115.3) $$\iint g(t,s)\, d\mu_n(t)\, d\mu_n(s) = \mathrm{I} + \mathrm{II} + \mathrm{III}$$

where

$$\mathrm{I} = \iint h(t)\, h(s)\, g(t,s)\, d\mu_n(t)\, d\mu_n(s)$$
$$\mathrm{II} = \iint \left(1-h(t)\right) h(s)\, g(t,s)\, d\mu_n(t)\, d\mu_n(s)$$
$$\mathrm{III} = \iint \left(1-h(s)\right) g(t,s)\, d\mu_n(t)\, d\mu_n(s)$$

We have that

$$|\mathrm{III}| \le \|g\|_\infty \int d\mu_n(t) \int (1-h(s))\, d\mu_n(s)$$

and so as $\mu_n \to \mu$

(116.1) $$\overline{\lim}_n |\mathrm{III}| \le \|g\|_\infty \int_{|s|>M} d\mu(s) < \varepsilon \|g\|_\infty$$

Similarly

(116.2) $$\overline{\lim}_n |\mathrm{II}| \le \varepsilon \|g\|_\infty$$

On the other hand, $\exists$ a polynomial $p(t,s) = \sum a_{ij} t^i s^j$

such that

(116.3) $$|p(t,s) - g(t,s)| \le \varepsilon \qquad \forall |t|, |s| \le M+1.$$

Hence

(116.4) $$|h(t)h(s)g(t,s) - h(t)h(s)p(t,s)| < \varepsilon \qquad \forall\, t, s.$$

Thus

$$\mathrm{I} = \iint h(t)h(s)\, p(t,s)\, d\mu_n(t)\, d\mu_n(s)$$

$$+ \iint h(t)h(s)\,(g(t,s) - p(t,s))\, d\mu_n(t)\, d\mu_n(s)$$

$$= \mathrm{I}' + \mathrm{II}'$$

By (116.3), $|\mathrm{II}'| \le \varepsilon$. Also

$$\mathrm{I}' = \sum_{i,j} a_{ij} \left(\int h(t)\, t^i\, d\mu_n(t)\right)\left(\int h(s)\, s^j\, d\mu_n(s)\right)$$

$$\xrightarrow[n\to\infty]{} \sum a_{ij} \Big(\underbrace{\int_{\Lambda} t^i \, d\mu(t)}_{h(t)}\Big)\Big(\underbrace{\int_{\Lambda} s^j \, d\mu(s)}_{h(s)}\Big)$$

$$= \iint h(t)\, h(s)\, p(t,s)\, d\mu(s)\, d\mu(t)$$

$$= \iint h(t)\, h(s)\, g(t,s)\, d\mu(t)\, d\mu(s) + O(\varepsilon)$$

$$= \iint g(t,s)\, d\mu(t)\, d\mu(s) \quad + O(\varepsilon) + O(2\varepsilon \|g\|_\infty).$$

As $\varepsilon > 0$ is arbitrary, it follows from the above calculations that as $n \to \infty$

$$\iint g(t,s)\, d\mu_n(t)\, d\mu_n(s) \to \iint g(t,s)\, d\mu(t)\, d\mu(s)$$

Applying this result to (115.1) we find for any $L > 0$

$$\varliminf_n H(\mu_n) \geq \iint \min(L, k(t,s))\, d\mu(t)\, d\mu(s)$$

Letting $L \uparrow \infty$, we finally obtain (115.0) by the monotone convergence Th^m (note: $\min(L, k(t,s)) \geq \min(L, C_v) = 0$)

Now choose a sequence $\{\mu_n\}$ in $M_1(\mathbb{R})$ such that

(117.1) $$H(\mu_n) \leq E^v + \frac{1}{n}$$

By (114.2)

$$(118.1) \qquad E^V + \frac{1}{n} \geq H(\mu_n) \geq \frac{1}{2} \iint \big(\psi(s) + \psi(t)\big)\, d\mu_n(s)\, d\mu_n(t)$$

$$= \int \psi(t)\, d\mu_n(t)$$

But for any b, $\exists\, M$ st $\psi(t) > b$ for $|t| \geq M$.
Thus as $\psi(t) \geq c_V \geq 0$,

$$E^V + 1 \geq E^V + \frac{1}{n} \geq \int_{|t| \geq M} \psi(t)\, d\mu_n(t) \geq b \int_{|t| \geq M} d\mu_n(t).$$

from which it follows that $\{\mu_n\}$ is tight.

Hence $\exists\ \mu \in M_1(\mathbb{R})$ and a subsequence $\{\mu_{n(k)}\}$ st $\mu_{n(k)} \to \mu$. But then from (115.0), (118.1)

$$E^V \leq H(\mu) \leq \varliminf_{k \to \infty} H(\mu_{n(k)}) \leq E^V$$

and so

$$(118.2) \qquad H(\mu) = E^V$$

Thus the infimum in (113.4) is attained. We will consider the uniqueness of the minimizer μ below, but we show first that any $\mu' \in M_1(\mathbb{R})$ st $H(\mu') = E^V$ necessarily has compact support.

So let $H(\mu') = E^V$ for some $\mu' \in M_1(\mathbb{R})$,

and let D be any Borel set in $\mathbb{R}$.

Define

(119.1) $$\mu'_\varepsilon = \frac{\mu' + \varepsilon \mu' \restriction_D}{1 + \varepsilon \mu'(D)}, \quad |\varepsilon| < 1.$$

where $\mu' \restriction_D$ is the restriction of μ' to D.

Clearly $\mu'_\varepsilon \in M_1(\mathbb{R})$ for $|\varepsilon| < 1$. We have

(119.2) $$H(\mu_\varepsilon) = \frac{1}{(1+\varepsilon\mu'(D))^2} \int k(t,s)\, d(\mu' + \varepsilon\mu'\restriction_D)(t)\, d(\mu' + \varepsilon\mu'\restriction_D)(s)$$

$$= \frac{H(\mu')}{(1+\varepsilon\mu'(D))^2} + \frac{2\varepsilon}{(1+\varepsilon\mu'(D))^2} \int k(t,s)\, d\mu'\restriction_D(t)\, d\mu'(s)$$

$$+ \frac{\varepsilon^2}{(1+\varepsilon\mu'(D))^2} \int k(t,s)\, d\mu'\restriction_D(t)\, d\mu'\restriction_D(s)$$

Note that $k(t,s) \geq c_V \geq 0$ and $\iint k(t,s)\, d\mu'(t)\, d\mu'(s) = E^V < \infty$, and hence

$k(t,s)\chi_D(t)$ and $k(t,s)\chi_D(t)\chi_D(s)$ are integrable w.r.t.

$d\mu'(t) \otimes d\mu'(s)$, and hence all the terms in (119.2) are finite and the formula is valid.

But by the minimal property of μ', we must have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} H(\mu'_\varepsilon) = 0$$

Thus

$$2\int k(t,s)\, d\mu'|_D(t)\, d\mu'(s) - 2H(\mu')\,\mu'(D) = 0$$

ie

$$\iint \big(k(t,s) - H(\mu')\big)\, d\mu'|_D(t)\, d\mu'(s) = 0 \quad ,$$

and hence by (114.2)

(120.1) $$0 \geq \iint \left(\tfrac{1}{2}\psi(t) + \tfrac{1}{2}\psi(s) - H(\mu')\right) d\mu'\Gamma_D(t)\, d\mu'(s)$$

$$= \frac{1}{2}\int \left[\psi(t) + \left(\int \psi(s)\, d\mu'(s)\right) - 2H(\mu')\right] d\mu'\Gamma_D(t)$$

But by (112.4), $\exists$ M st

$$\psi(t) - 2H(\mu') + \int \psi(s)\, d\mu'(s) \geq 1$$

for $|t| \geq M$ (Note that as $\infty > E^V = H(\mu') =$

$$= \iint k(t,s)\, d\mu'(t)\, d\mu'(s) \geq \frac{1}{2}\iint (\psi(t) + \psi(s))\, d\mu'(t)\, d\mu'(s)$$

$$= \int \psi(t)\, d\mu'(t),$$

$\int \psi(t)\, d\mu'(t)$ is a finite # greater or equal to c_V)

Hence if $\mu(D \setminus \{|x| \leq M\}) > 0$, (120.1) gives a contradiction):