$0 < \mu'(\mathbb{R} \setminus \{|x| \le M\}) \le 0$. It follows that

(121.1) $$\operatorname{supp} \mu' \subset [-M, M]$$

Note that since $V(x) \ge V_0 > -\infty$, and $d\mu'$ has compact support,

$$-\infty < V_0 \le \int V(x)\, d\mu'(x) < \infty$$

and hence

(121.2) $$\infty > E^V - V_0 > \iint \log|t-s|^{-1}\, d\mu'(t)\, d\mu'(s) > -\infty$$

We say that $d\mu'$ has <u>finite logarithmic</u> <u>and</u> <u>potential</u> <u>energy</u>.

We now consider uniqueness. Recall that a finite signed measure μ has a (unique) Jordan decomposition $\mu = \mu^+ - \mu^-$ where $\mu^{\pm}$ are positive finite measures, $0 \le \int d\mu^{\pm} < \infty$, and are mutually singular i.e. supported on disjoint sets. If μ_1 and μ_2 are 2 finite measures,

$\mu = \mu_1 - \mu_2$ is a finite signed measure. We say that a real or complex valued function f is integrable if it is integrable with respect to $|\mu| = \mu^+ + \mu^-$, and in this case

$$\int f\,d\mu = \int f\,d\mu^+ - \int f\,d\mu^-$$

If $\mu = \mu_1 - \mu_2$ is any decomposition of the finite signed measure μ as a difference of finite measures, then it is easy to see that

$$\int f\,d\mu = \int f\,d\mu_1 - \int f\,d\mu_2$$

for all f that are integrable with respect to $d\mu_1$ and $d\mu_2$ (and hence integrable wrt $d\mu$).

Lemma 122.1

Let μ be a finite signed measure on $\mathbb{R}$ with mean zero i.e. $\int d\mu = 0$ and with compact support. Then

(122.2) $$\iint \log|x-y|^{-1}\,d\mu(x)\,d\mu(y) \geq 0 .$$

Remark:

Inequality (122.2) needs some interpretation. Although the function $\log |x-y|^{-1}$ is bded below on the compact support of $d\mu(x) \otimes d\mu(y)$, the function is not bded above. Therefore we do not know whether it is integrable wrt

$$d\mu(x) \otimes d\mu(y) = \left(d\mu_1(x) \otimes d\mu_1(y) + d\mu_2(x) \otimes d\mu_2(y)\right)$$

$$- \left(d\mu_1(x) \otimes d\mu_2(y) + d\mu_2(x) \otimes d\mu_1(y)\right)$$

where $d\mu = d\mu_1 - d\mu_2$ is any decomposition of $d\mu$ as a difference of measures. So what we mean by (122.2) is that for any decomposition $d\mu = d\mu_1 - d\mu_2$, where $d\mu_1, d\mu_2$ are measures with compact support,

(123.1) $$\iint \log |x-y|^{-1} \left(d\mu_1(x)\, d\mu_1(y) + d\mu_2(x)\, d\mu_2(y)\right)$$

$$\geq \iint \log |x-y|^{-1} \left(d\mu_1(x)\, d\mu_2(y) + d\mu_2(x)\, d\mu_1(y)\right)$$

$$= 2 \iint \log |x-y|^{-1}\, d\mu_1(x)\, d\mu_2(y)$$

$$= 2 \iint \log |x-y|^{-1}\, d\mu_2(x)\, d\mu_1(y)$$

Here there is no ambiguity ~~b~~ since $\log |x-y|^{-1}$ is bded below on the compact set $\operatorname{supp}\mu \times \operatorname{supp}\mu$ and hence both sides of

the inequality in (123.1) are well defined.
Of course if $\log|x-y|^{-1}$ is integ. wrt both the measures

$d\mu_1(x)\, d\mu_1(y)$ and $d\mu_2(x)\, d\mu_2(y)$, then it is integrable

wrt both the measures $d\mu_1(x)\, d\mu_2(y)$ and $d\mu_2(x)\, d\mu_1(y)$.

In this case $|x-y|^{-1}$ is integ. with respect to $d\mu(x) \otimes d\mu(y)$

and so (122.2) is true as it stands.

<u>Proof of Lemma (122.1)</u>: For any real $\varepsilon > 0$

$$\log(s^2+\varepsilon^2) = \log \varepsilon^2 + \int_0^s \frac{2t}{t^2+\varepsilon^2}\, dt$$

$$= \log \varepsilon^2 + 2\, \mathrm{Im} \int_0^s \frac{i}{t+i\varepsilon}\, dt$$

$$= \log \varepsilon^2 + 2\, \mathrm{Im} \int_0^s dt \int_0^\infty e^{i(t+i\varepsilon)u}\, du$$

$$= \log \varepsilon^2 + 2\, \mathrm{Im} \int_0^\infty du\, e^{-\varepsilon u} \int_0^s dt\, e^{itu}$$

$$= \log \varepsilon^2 + 2\, \mathrm{Im} \int_0^\infty du\, e^{-\varepsilon u}\, \frac{e^{itu}}{iu}\Big|_0^s$$

$$= \log \varepsilon^2 + 2\, \mathrm{Im} \int_0^\infty du\, e^{-\varepsilon u}\, \frac{e^{ius}-1}{iu}$$

Hence for the function $\log(|x-y|^2+\varepsilon^2)$, which is clearly integrable with respect to the (compactly supported measure) $d\mu \otimes d\mu$,

$$\iint \log(|x-y|^2+\varepsilon^2)\, d\mu(x)\, d\mu(y)$$

$$= \iint \log\varepsilon^2\, d\mu(x)\, d\mu(y) + 2\,\mathrm{Im} \int_0^\infty du\, e^{-\varepsilon u} \iint d\mu(x)\, d\mu(y) \frac{e^{i(x-y)u}-1}{iu}$$

$$= 2\,\mathrm{Im} \int_0^\infty du \frac{e^{-\varepsilon u}}{iu} |\hat{\mu}(u)|^2 \qquad (\text{as } \int d\mu = 0)$$

$$= -2 \int_0^\infty du\, e^{-\varepsilon u} \frac{|\hat{\mu}(u)|^2}{u}, \quad \text{where } \hat{\mu}(u) = \int e^{-ixu} d\mu(x).$$

Thus

(125.1) $$-\iint \log(|x-y|^2+\varepsilon^2)\, d\mu(x)\, d\mu(y) = 2\int_0^\infty du\, e^{-\varepsilon u} \frac{|\hat{\mu}(u)|^2}{u} \geq 0.$$

Note That $\hat{\mu}(0) = \int d\mu(x) = 0$, and as $\hat{\mu}(u)$ is analytic in u, there is no singularity in the integral on the RHS as $u \to 0$. Writing out (125.1) we find for any decomposition $\mu = \mu_1 - \mu_2$, where μ_1, μ_2 are measures with compact support

$$\iint \log(|x-y|^2+\varepsilon^2)^{-\frac{1}{2}} (d\mu_1(x)\, d\mu_1(y) + d\mu_2(x)\, d\mu_2(y))$$

$$= \iint \log(|x-y|^2+\varepsilon^2)^{-\frac{1}{2}} (d\mu_1(x)\, d\mu_2(y) + d\mu_2(x)\, d\mu_1(y))$$

$$+ \int_0^\infty du\, e^{-\varepsilon u} \frac{|\hat{\mu}(u)|^2}{u}$$

Letting $\varepsilon \downarrow 0$, by the monotone convergence th^m

(we use here that $\log((|x-y|^2+\varepsilon^2)^{-\frac{1}{2}}$ is bded below uniformly in $\varepsilon<1$ (by, for example, $\log((|x-y|^2+1)^{-1/2}$ which is integrable as the measures have compact support.)

$$(126.1) \quad \iint \log|x-y|^{-1}\left(d\mu_1(x)\,d\mu_1(y) + d\mu_2(x)\,d\mu_2(y)\right)$$

$$= \iint \log|x-y|^{-1}\left(d\mu_1(x)\,d\mu_2(y) + d\mu_2(x)\,d\mu_1(y)\right)$$

$$+ \int_0^\infty \frac{|\hat{\mu}(u)|^2}{u}\,du$$

which implies, in particular, inequality (122.2)

Now suppose $\mu, \tilde{\mu} \in \mathcal{M}_1(\mathbb{R})$ are two prob. distrib's for which $E^V = H(\mu) = H(\tilde{\mu}) = \inf_{\mu' \in \mathcal{M}_1(\mathbb{R})} H(\mu')$.

By the argument above, μ and $\tilde{\mu}$ have compact support. As $-\infty < E^V < \infty$, it follows as before (see (121.21)) that $\log|x-y|^{-1}$ is integrable with respect to $d\mu \otimes d\mu$ and $d\tilde{\mu} \otimes d\tilde{\mu}$. Hence it follows from (123.1) that $\log|x-y|^{-1}$ is integrable with respect to $d\mu \otimes d\tilde{\mu}$, and $d\tilde{\mu} \times d\mu$: all we need to note is that $\int d\mu - d\tilde{\mu} = 1 - 1 = 0$. It follows that $\log|x-y|^{-1}$ is integrable wrt to the measure $d\mu_t \otimes d\mu_t$ where

$$\mu_t \equiv t\tilde{\mu} + (1-t)\mu \tag{127.1}$$

$$= \mu + t(\tilde{\mu} - \mu), \qquad t \in \mathbb{R}.$$

Furthermore

$$f(t) = \iint \log|x-y|^{-1}\, d\mu_t(x)\, d\mu_t(y) + \int V d\mu + t \int V d(\tilde{\mu} - \mu)$$

$$= \iint \log|x-y|^{-1}\, d\mu(x)\, d\mu(y) + \int V d\mu$$

$$+ 2t \iint \log|x-y|^{-1}\, d\mu(x)\, d(\tilde{\mu}-\mu)(y) + t \int V d(\tilde{\mu}-\mu)$$

$$+ t^2 \iint \log|x-y|^{-1}\, d(\tilde{\mu}-\mu)(x)\, d(\tilde{\mu}-\mu)(y)$$

(Note by the above discussion all the integrals are well-defined and finite.). Now $\tilde{\mu} - \mu$ has mean zero & compact support. Hence by (122.2), $f(t)$ is a convex function of t, and so, for $0 \le t \le 1$,

$$H(\mu_t) = f(t) = f(t\cdot 1 + (1-t)\cdot 0) \le t f(1) + (1-t) f(0) = t H(\tilde{\mu}) + (1-t) H(\mu)$$

$= E^V$ as $H(\mu) = H(\tilde{\mu}) = E^V$.

Thus $H(\mu_t) = E^V = \text{const.}$ In particular, by (126.1),

$$0 = \frac{1}{2} f''(t) = \iint \log|x-y|^{-1} \, d(\tilde{\mu}-\mu)(x) \, d(\tilde{\mu}-\mu)(y)$$

$$= \int_0^\infty \frac{|\widehat{(\tilde{\mu}-\mu)}(u)|^2}{u} \, du$$

Thus $\hat{\tilde{\mu}}(u) = \hat{\mu}(u)$ $\forall u > 0$, and hence $\forall u$

as $\hat{\tilde{\mu}}(u) = \overline{\hat{\tilde{\mu}}(-u)}$, $\hat{\mu}(u) = \overline{\hat{\mu}(-u)}$

(alternatively $\hat{\tilde{\mu}}(u)$ and $\hat{\mu}(u)$ are entire!). Thus

$\tilde{\mu} = \mu$ as desired. This completes the proof of Th^m 113.3.

Remark: The proof of Th^m 113.3 follows the "standard" path for the solution of a convex minimization problem.

Recall the definition (92.1) of the correlation function $R_1(x)$:

$$(128.1) \qquad R_1(x) = n \int P_N(x, x_2, \ldots, x_N) \, dx_2 \ldots dx_N$$

where

(129.1) $$P_N(x_1, x_2, \dots, x_N) = \frac{1}{Z_N} e^{-N\sum_{i=1}^N V(x_i)} \prod_{1\le i<j\le N} (x_i - x_j)^2 \, d^N x$$

and Z_N is the normalization factor (partition function)

(129.2) $$Z_N = \int_{\mathbb{R}^N} e^{-N\sum_1^N V(x_i)} \prod_{i<j} (x_i - x_j)^2 \, d^N x .$$

Our goal is, eventually, to prove the following result.

<u>Th^m 129.3</u> Let $V(x)$ be a continuous function satisfying (112.1), $\frac{V(x)}{\log(x^2+1)} \to +\infty$ as $|x| \to \infty$. Then

(129.4) $$\frac{1}{N} R_1(x)\,dx \to d\mu^{ed}$$

as $N \to \infty$. □

By (98.2), $\int_\Omega R_1(x)\,dx = \mathbb{E}\mathrm{xp}\,(\#\{\text{eigenvalues in } \Omega\})$, $\Omega \subset \mathbb{R}$. Thus (129.4) says that

$$\mathbb{E}\mathrm{xp}\,(\#\{\text{eigenvalues in } \Omega\}/N) \to \mu^{ed}(\Omega)$$

In other words, $d\mu^{ed}$ is the <u>density of states</u>.

Recall from (113.1), $k(t,s) = \log|t-s|^{-1} + \frac{1}{2}V(t) + \frac{1}{2}V(s)$.

(130.1) Let
$$K_N(x) = \sum_{1\le i\ne j\le N} k(x_i, x_j)$$
$$= \sum_{1\le i\ne j\le N} \log|x_i - x_j|^{-1} + (N-1)\sum_{i=1}^{N} V(x_i)$$

The key to the proof of (129.4) is the following "large deviation" estimate!

Lemma 130.2 Let

$$P_N(x)\,dx = \frac{1}{Z_N} e^{-N\sum_1^N V(x_i)} \prod_{i<j} |x_i - x_j|^2 \, d^N x$$

be a prob. meas. (as in (129.1) above) and let $\eta > 0$ be given. Set

(130.3) $$A_{N,\eta} = \{x \in \mathbb{R}^N : \frac{1}{N^2} K_N(x) \le E^V + \eta\}$$

and let $a \ge 0$ be any positive number. Then $\exists$ $N^* = N^*(\eta)$ which depends on η, but not on a, st

(130.4) $$P_N(\mathbb{R}^N \setminus A_{N,\eta+a}) \le e^{-aN^2}, \quad N \ge N^*(\eta)$$

Proof: For any $\varepsilon' > 0$, set

(130.5) $$\psi_{\varepsilon'}(x) = \frac{1}{2\varepsilon'} \int_{x-\varepsilon'}^{x+\varepsilon'} d\mu^{eq}(t)$$

where μ^{eq} is the eqm. meas. for V. As $H(\mu^{eq}) < \infty$, μ^{eq} has no point masses (why?), and hence $\psi_{\varepsilon'}(x) \geq 0$ is a continuous function of x. In addition, as μ^{eq} has compact support, so does $\psi_{\varepsilon'}(x)$. Also $\psi_\varepsilon(x)$ is the convolution of μ^{eq} with a function of mean 1 and so $\int \psi_\varepsilon(x)\,dx = 1$. An elementary argument shows that $\psi_{\varepsilon'}(x)\,dx \to \mu^{eq}$ as $\varepsilon' \downarrow 0$. We want to show more, viz., as $\varepsilon' \downarrow 0$

(131.1) $$H(\psi_{\varepsilon'}) \to H(\mu^{eq}) = E^V$$

Interchanging the order of integration, and then rescaling, we obtain

(131.2) $$\iint \log|t-s|\, \psi_{\varepsilon'}(t)\, \psi_{\varepsilon'}(s)\, dt\, ds$$

$$= \iint \mu^{eq}(t)\, \mu^{eq}(s)\, H_{\varepsilon'}(t,s)$$

where

(131.3) $$H_{\varepsilon'}(t,s) = \frac{1}{4\varepsilon'^2} \int_{-\varepsilon'}^{\varepsilon'} \int_{-\varepsilon'}^{\varepsilon'} \log|x-y+t-s|\; dx\, dy$$

$$= \frac{1}{4\varepsilon'^2} \int_{-\varepsilon'}^{\varepsilon'}\int_{-\varepsilon'}^{\varepsilon'} \log|x-y+|t-s|| \,dx\,dy \,, \quad \text{by symmetry}$$

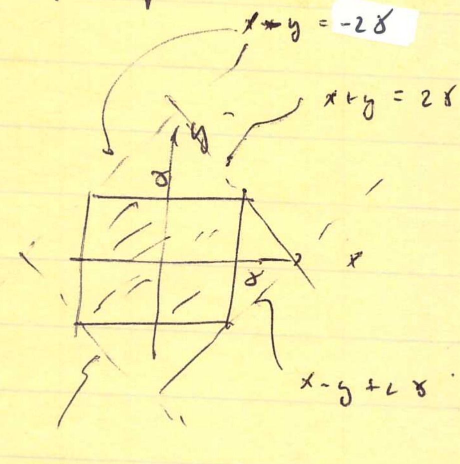
$$= \log|t-s| + \frac{1}{4\gamma^2} \int_{-\gamma}^{\gamma}\int_{-\gamma}^{\gamma} \log|1+(x-y)| \,dx\,dy$$

where $\gamma = \varepsilon'/|t-s|$, $t \neq s$. The above integral is clearly bdded by

$$\frac{1}{4\gamma^2} \iint_{\substack{|x-y|\le 2\gamma \\ |x+y|\le 2\gamma}} \left|\log|1+(x-y)|\right| dx\,dy \,.$$

$$= \frac{1}{8\gamma^2} \iint_{\substack{|u|<2\gamma \\ |v|<2\gamma}} \left|\log|1+u|\right| du\,dv$$

$$= \frac{1}{2\gamma} \int_{-2\gamma}^{2\gamma} \left|\log|1+u|\right| du$$



which is bounded in turn, using simple estimates (exercise!), by $c\log(1+\gamma)$ for some $c>0$

Thus from (131.2) (131.3)

$$\left| \iint \log|t-s| \, \psi_{\varepsilon'}(t)\psi_{\varepsilon'}(s)\,dt\,ds - \iint \log|t-s| \, q\mu^{ed}(t)\, q\mu^{ed}(s) \right|$$

$$\le c \int \log\left(1 + \frac{\varepsilon'}{|t-s|}\right) q\mu^{ed}(t)\, q\mu^{ed}(s)$$

$$= c\left[\iint \log(|t-s|+\varepsilon')\, d\mu^{ed}(t)\, d\mu^{ed}(s) - \iint \log|t-s|\, d\mu^{ed}(t)\, d\mu^{ed}(s) \right].$$

As $\log(|t-s|+1)$ is integrable ~~below~~ on the (compact) support of $d\mu^{ed}(t) \otimes d\mu^{ed}(s)$, we conclude by monotone convergence that $\iint \log|t-s|^{-1} \psi_{\varepsilon'}(t)\, \psi_{\varepsilon'}(s)\, dt\, ds \to \iint \log|t-s|^{-1} d\mu^{ed}(t)\, d\mu^{ed}(s)$ as $\varepsilon' \downarrow 0$ (Note: we have by (121.2), for example, $\left|\iint \log|t-s|^{-1} d\mu^{ed}(t)\, d\mu^{ed}(s)\right| < \infty$). Thus $H(\psi_{\varepsilon'}) \to H(\mu^{ed})$ as $\int V(s)\, \psi_{\varepsilon'}(s)\, ds \to \int V(s)\, d\mu^{ed}(s)$.

In particular, we conclude for any given $\varepsilon > 0$, $\exists\, \varepsilon' = \varepsilon'(\varepsilon)$ st

(133.1) $$H(\phi_\varepsilon) \le E^V + \varepsilon/2$$

where

(133.2) $$\phi_\varepsilon = \psi_{\varepsilon'} = \psi_{\varepsilon'(\varepsilon)}$$

We now use the above calculations to derive a lower bound on Z_N. Let

(133.3) $$E^N = \{x \in \mathbb{R}^N : \prod_{i=1}^{N} \phi_\varepsilon(x_i) > 0\}.$$

Then

$$(134.1) \quad Z_N = \int_{\mathbb{R}^N} e^{-N\sum_1^N V(x_i)} \prod_{1\le i<j\le N} |x_i - x_j|^2 \, d^N x$$

$$= \int_{\mathbb{R}^N} e^{-K_N(x) - \sum_1^N V(x_i)}$$

$$\geq \int_{E^N} e^{-K_N(x) - \sum_1^N V(x_i)}$$

$$= \int_{E^N} e^{-K_N - \sum V(x_i) + \sum_1^N [-\log \phi_\varepsilon(x_i)]} \times \prod_{i=1}^N \phi_\varepsilon(x_i) \, d^N x$$

By Jensen's inequality

$$\int e^{f(x)} d\mu(x) \geq e^{\int f(x)\, d\mu(x)}$$

$\forall f \in L^1(d\mu)$, $\int d\mu = 1$, and from (134.1), we obtain

$$\log Z_N \geq - \int K_N(x) \prod_1^N \phi_\varepsilon(x_i)\, d^N x$$

$$- \int \sum V(x_i) \prod_1^N \phi_\varepsilon(x_i)\, d^N x$$

$$- \int \Big(\sum_1^N \log \phi_\varepsilon(x_i)\Big) \prod_1^N \phi_\varepsilon(x_i)\, d^N x$$

$$= -N(N-1)\, H(\phi_\varepsilon) - N \int V(t)\, \phi_\varepsilon(t)\, dt - N \int (\log \phi_\varepsilon(t))\, \phi_\varepsilon(t)\, dt$$

$$\geq -N^2 H(\phi_\varepsilon) - C_\varepsilon N \geq -N^2 (E^V + \varepsilon/2) - C_\varepsilon N$$

for some constant c_ε. Thus

(135.1) $$\frac{1}{N^2} \log Z_N \geq -(E^V + \varepsilon)$$

for N suff. large, say $N \geq N_1$. Note that N_1 depends only on ε, $N_1 = N_1(\varepsilon)$.

Now by (135.1), for $N \geq N_1$,

$$P_N(\mathbb{R}^N \setminus A_{N, m+a})$$

$$= \frac{1}{Z_N} \int_{\{K_N(x) > N^2(E^V + m + a)\}} e^{-K_N - \sum_1^N V(x_i)} \, d^N x$$

$$\leq \int_{\mathbb{R}^N} e^{-\sum_1^N V(x_i)} e^{(N^2[-(E^V + m + a) + E^V + \varepsilon)]} \, d^N x$$

$$= \left(\int e^{-V(t)} dt\right)^N e^{N^2(\varepsilon - m - a)}$$

$$\leq \left(\int e^{-V(t)} dt\right)^N e^{-aN^2} e^{-\frac{m}{2} N^2}$$

provided we choose $\varepsilon < m/2$. Thus for N suff. large, $N \geq N_2 = N_2(m) \geq N_1$, we have $P_N(\mathbb{R}^N \setminus A_{N, m+a}) \leq e^{-aN^2}$, we proves Lemma 130.2'. $\square$

$\Rightarrow$ The next lemma is the main technical result that is used to prove the convergence $\frac{1}{N} R_N(x) dx \to \psi^{eq}(x)$.

Lemma 136.1 Let $\Phi : \mathbb{R}^k \to \mathbb{R}$ be bounded and continuous.

Then

(136.2)
$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}xp_N \left(e^{-\frac{1}{N^{k-1}} \sum_{i_1,\dots,i_k} \Phi(x_{i_1},\dots,x_{i_k})} \right)$$

$$= \int_{\mathbb{R}^k} \Phi(t_1,\dots,t_k)\, d\mu^{eq}(t_1) \dots d\mu^{eq}(t_k).$$

where $\mathbb{E}xp_N$ denotes expectation wrt $P_N(x)\, d^N x$.

Proof: Let $\eta > 0$ be given and let $\chi = \chi_{A_{N,2\eta}}$ be the characteristic function of the set $A_{N,2\eta}$ (see previous lemma). Now

$$\overline{\lim_N} \frac{1}{N} \log \mathbb{E}xp_N \left(e^{N^{-k+1} \sum_{i_1,\dots,i_k} \Phi(x_{i_1},\dots,x_{i_k})} \right)$$

$$= \overline{\lim_N} \frac{1}{N} \log \mathbb{E}xp_N \left(e^{N^{-k+1} \sum_{i_1,\dots,i_k} \Phi(x_{i_1},\dots,x_{i_k})} \chi + e^{N^{-k+1} \sum_{i_1,\dots,i_k} \Phi(x_{i_1},\dots,x_{i_k})} (1-\chi) \right)$$

$$= \overline{\lim_N} \left(\frac{1}{N} \log \mathbb{E}xp_N \left(e^{N^{-k+1} \sum_{i_1,\dots,i_k} \Phi(x_{i_1},\dots,x_{i_k})} \chi \right) + \frac{1}{N} \log \left(1 + \frac{\mathbb{E}xp_N \left((1-\chi) e^{\Phi} \right)}{\mathbb{E}xp_N \left(\chi e^{\Phi} \right)} \right) \right)$$

where

(137.1) $$\Phi = N^{-k+1} \sum_{i_1,\dots,i_k} \phi(x_{i_1}, \dots, x_{i_k}) .$$

As ϕ is bded, say $|\phi| \le b$,

$$|\Phi| \le N^{-k+1}\, b\, N^k = bN .$$

Hence

$$\frac{\mathbb{E}xp_N\left((1-\chi)e^{\Phi}\right)}{\mathbb{E}xp_N\left(\chi e^{\Phi}\right)} \le \frac{\mathbb{E}xp_N\left((1-\chi)e^{bN}\right)}{\mathbb{E}xp_N\left(\chi e^{-bN}\right)}$$

$$= e^{2bN} \frac{\mathbb{E}xp_N(1-\chi)}{1-\mathbb{E}xp_N(1-\chi)}$$

$$\le e^{2bN} \frac{e^{-\eta N^2}}{1-e^{-\eta N^2}}$$

for $N \ge N^*$ by 130.2. Hence

(137.1) $$\lim_{N\to\infty} \frac{1}{N} \log\left(1 + \frac{\mathbb{E}xp_N\left((1-\chi)e^{\Phi}\right)}{\mathbb{E}xp_N\left(\chi e^{\Phi}\right)}\right) = 0,$$

and so

(137.2) $$\overline{\lim_N} \frac{1}{N} \log \mathbb{E}xp_N\, e^{\Phi} = \overline{\lim_N} \frac{1}{N} \log \mathbb{E}xp_N\left(e^{\Phi}\chi\right)$$

Now the set $A_{N,2\eta}$ is compact. Indeed