Indeed if $x \in A_{N,2m}$, then by (114.2), $k \geq \frac{1}{2}\psi(s) + \frac{1}{2}\psi(t)$,

$$(138.1) \quad E^V + 2m \geq \frac{1}{N^2} K_N(x) = \frac{1}{N^2} \sum_{i \neq j} k(x_i, x_j)$$

$$\geq \frac{1}{N^2} \sum_{i \neq j} \left(\frac{1}{2}\psi(x_i) + \frac{1}{2}\psi(x_j) \right)$$

$$= \frac{N-1}{N^2} \sum_{1}^{N} \psi(x_i)$$

$$\geq \frac{N-1}{N^2} \left((N-1) c_V + \psi(x_j) \right)$$

for any $j = 1, \ldots, N$. As $\psi(x) \to +\infty$ as $|x| \to \infty$ (see (112.41)), we conclude that $|x_j|$ lies in a bded set for each j. It is easy to see (exercise) that $A_{N,2m}$ is also closed, and hence, compact. It follows that the cont. function Φ achieves its maximum at some point $x^* \in A_{N,2m}$. Thus by (137.2),

$$\overline{\lim_N} \frac{1}{N} \log \mathbb{E}xp_N (e^{\Phi})$$

$$= \overline{\lim_N} \frac{1}{N} \log \mathbb{E}xp_N (e^{\Phi} \chi) \leq \overline{\lim_N} \frac{1}{N} \log e^{\Phi(x^*)} (1 - \underbrace{\mathbb{E}xp_N (1-\chi)}_{O(e^{-N^2 m})})$$

$$= \overline{\lim_N} \frac{1}{N} \Phi(x^*)$$

ie

(139.0) $$\overline{\lim_N} \frac{1}{N} \log \operatorname{Exp}_M e^{\Phi} \le \overline{\lim_N} \int \phi(t_1, \dots, t_k)\, d\nu_N(t_1) \dots d\nu_N(t_k).$$

where

(139.1) $$\nu_N(t) = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j^*}, \quad x^* = (x_1^*, \dots, x_N^*).$$

Since $x^* \in A_{N, 2m}$, we again have as in (138.1),

(139.2) $$(N-1) \sum_{i=1}^{N} \psi(x_i^*) \le K_N(x^*) \le N^2 (E^\nu + 2m)$$

Thus

(139.3) $$\int \psi(t)\, d\nu_N(t) \le \frac{N}{N-1} (E^\nu + 2m) \le c$$

which implies as before (cf p118) that $\{\nu_N\}$ is tight.

Now we first choose a subsequence N_j st

$$\int \phi(t_1, \dots, t_k)\, d\nu_{N_j}(t_1) \dots d\nu_{N_j}(t_k) \to \overline{\lim_N} \int \phi(t_1, \dots, t_k)\, d\nu_N(t_1) \dots d\nu_N(t_k)$$

We then choose by tightness a further subsequence $\{\nu_{N_{j(i)}}\}$ which is convergent, $\nu_{N_{j(i)}} \to \nu^*$ for some $\nu^* \in M_1(R)$, and

(140.1) and we still have

$$\int \phi(t_1,\dots,t_k)\, d\nu_{N_{j(i)}}(t_1)\dots d\nu_{N_{j(i)}}(t_k)$$

$$\to \overline{\lim_N} \int \phi(t_1,\dots,t_k)\, d\nu_N(t_1)\dots d\nu_N(t_k).$$

Note that ν^* depends on η, $\nu^* = \nu^*_\eta$.

Now for any L, $\min(L, k(t,s))$ is a continuous function and we have from (138.1)

$$E^\nu + 2\eta \geq \frac{K_{N_{j(i)}}(x^*)}{N^2_{j(i)}} = \frac{1}{N^2_{j(i)}} \sum_{i\neq j} k(x_i^*, x_j^*)$$

$$\geq \frac{1}{N^2_{j(i)}} \sum_{i\neq j} \min(L, k(x_i^*, x_j^*))$$

$$= \frac{1}{N^2_{j(i)}} \Big[\sum_{i,j} \min(L, k(x_i^*, x_j^*)) - L N_{j(i)} \Big]$$

so that

(140.2)
$$\iint \min(L, k(\tau,\sigma))\, d\nu_{N_{j(i)}}(\tau)\, d\nu_{N_{j(i)}}(\sigma) \leq E^\nu + 2\eta + \frac{L}{N_{j(i)}}.$$

Letting $N_{j(i)} \to \infty$ and, then $L \to \infty$, we find (140.3) $I(\nu^*) \leq E^\nu + 2\eta$.

Also, as before (cf pp 115 et seq), it follows from (139.0) and (140.1) that

(141.1) $$\overline{\lim_N} \frac{1}{N} \log \mathbb{E}xp_N \left(e^{\Phi} \right) \leq \int \phi(t_1, \ldots, t_k)\, d\nu^*(t_1) \cdots d\nu^*(t_k)$$

In addition from (138.1), $\int 4(t)\, d\nu_{N_{j(i)}}(t) \leq \dfrac{N_{j(i)}}{N_{j(i)}-1} \left(E^V + 2\eta \right)$

$\leq c$.

and so

(141.2) $$\int 4(t)\, d\nu^*(t) \leq c$$

Now as noted above $\nu^* = \nu^+_\eta$ and it follows as before from (141.2) that the measures ν^+_η, $\eta > 0$, are tight. Setting $\eta = \frac{1}{q}$ and letting $q \to \infty$ it follows by now familiar arguments that $\{ \nu^*_{1/q} \}$ converges weakly along a subsequence q_j to a prob. measure $\hat{\nu}$ satisfying

$$E^V \leq H(\hat{\nu}) \underset{\text{see (115.01)}}{\leq} \lim_{j \to \infty} H\left(\nu^+_{1/q_j} \right) \leq E^V \quad (\text{see (140.3)})$$

Hence $\hat{\nu} = \mu^{eq}$ and so $\nu^+_{1/q_j} \to \mu^{eq}$

Inserting this information into (141.1) we obtain finally that

(142.1) $$\overline{\lim_N} \frac{1}{N} \log \mathbb{E}xp_N \left(e^{\Phi} \right) \leq \int \phi(t_1, \ldots, t_k)\, d\mu^{ed}(t_1) \cdots d\mu^{ed}(t_k)$$

A similar argument, using points $x^{*'} \in A_{N,2\eta}$ at which $\sum \phi(x_{i_1}, \ldots, x_{i_k})$ achieves its minimum, shows that

(142.2) $$\underline{\lim_N} \frac{1}{N} \log \mathbb{E}xp_N \left(e^{\Phi} \right) \geq \int \phi(t_1, \ldots, t_k)\, d\mu^{ed}(t_1) \cdots d\mu^{e}(t_k)$$

This proves (136.2) and so Lemma (136.1).

The above calculations also prove the existence of the free energy.

Corollary 142.3

(Free energy Ξ)

(142.4) $$\exists \lim_{N \to \infty} -\frac{1}{N^2} \log Z_N = E^V = \iint \left(\log|t-s|^{-1} + V(t) \right) d\mu^{ed}(t)\, d\mu^{ed}(s)$$

Proof: From (135.1), $\frac{1}{N^2} \log Z_N \geq -(E^V + \varepsilon)$ ← (142.5)

for N suff. large. On the other hand

$$Z_N = \int \cdots \int e^{-K_N(x) - \sum_1^N V(x_i)}$$
$$\leq \int \cdots \int e^{-\sum_1^N V(x_i)}\, e^{-N(N-1) d_N^V}\, d^N x$$

where

(143.1) $$d_N^V = \frac{1}{N(N-1)} \inf_{x \in \mathbb{R}^N} K_N(x)$$

Thus

(143.2) $$Z_N \leq \left(\int e^{-V(t)} dt\right)^N e^{-N(N-1) d_N^V}$$

Hence

$$-\frac{1}{N^2} \log Z_N \geq -\frac{1}{N} \log \int e^{-V(t)} dt + \frac{N-1}{N} d_N^V.$$

Now (see ref (2) pp 145-147)

(143.3) $$\lim_{N \to \infty} d_N^V = E^V$$

and therefore

(143.4) $$\varliminf_{N} \left(-\frac{\log Z_N}{N^2}\right) \geq E^V$$

Together with (142.5), we obtain (142.4) and hence the Corollary.

We now prove the following corollary to Lemma 136.1.

<u>Corollary 143.5</u> For any bded, cont function $\phi : \mathbb{R}^k \to \mathbb{R}$, we have

(144.1) $$\lim_{N\to\infty} \frac{1}{N^k} \mathbb{E}xp_N \left(\sum_{i_1,\dots,i_k} \phi(x_{i_1},\dots,x_{i_k}) \right) = \int \phi(t_1,\dots,t_k)\, d\mu^{eq}(t_1)\cdots d\mu^{eq}(t_k).$$

Proof: For any bded cont. function $F: \mathbb{R}^n \to \mathbb{R}$,

observe that

$$f(t) = \log \mathbb{E}xp_N(e^{tF})$$

is a convex function of t. Indeed, $e^f = \mathbb{E}xp_N(e^{tF})$

and so

$$e^f f' = \mathbb{E}xp_N(Fe^{tF})$$

$$e^f (f')^2 + e^f f'' = \mathbb{E}xp_N(F^2 e^{tF})$$

Thus

$$e^f f'' = \mathbb{E}xp(F^2 e^{tF}) - \frac{(\mathbb{E}xp_N(Fe^{tF}))^2}{\mathbb{E}xp_N(e^{tF})}$$

But

$$0 < \mathbb{E}xp_N \left(\left(F - \frac{\mathbb{E}xp_N(Fe^{tF})}{\mathbb{E}xp_N(e^{tF})} \right)^2 e^{tF} \right)$$

$$= \mathbb{E}xp_N(F^2 e^{tF}) + \frac{(\mathbb{E}xp_N(Fe^{tF}))^2}{(\mathbb{E}xp_N(e^{tF}))} - 2\frac{(\mathbb{E}xp_N(Fe^{tF}))^2}{\mathbb{E}xp_N(e^{tF})}$$

$= e^f f''$ so we see that $f'' > 0$: in particular f is

/

convex. Convexity $\Rightarrow$ for $0 < t < 1$

(145.1) $$\frac{f(0) - f(-1)}{0 - (-1)} \le \frac{f(t) - f(0)}{t} \le \frac{f(1) - f(t)}{1 - t}$$

and letting $t \to 0$ we find

$$f(0) - f(-1) \le f'(0) \le f(1) - f(0)$$

As $f(0) = 0$, we see that for $F = n^{-k+1} \sum_{i_1, \dots, i_k} \phi(x_{i_1}, \dots, x_{i_k})$
$= \Phi$

$$-\log \mathbb{E}xp_N (e^{-\Phi}) \le \mathbb{E}xp_N(\Phi) \le \log \mathbb{E}xp_N e^{\Phi}$$

But then from (136.2)

$$-\int_{\mathbb{R}^k} -\phi(t_1, \dots, t_k) \, d\mu^{eq}(t_1) \dots d\mu^{eq}(t_k)$$

$$\le \varliminf_{N \to \infty} \mathbb{E}xp_N \frac{1}{N} \Phi \le \varlimsup_{N \to \infty} \frac{1}{N} \mathbb{E}xp_N \Phi$$

$$\le \int \phi(t_1, \dots, t_k) \, d\mu^{eq}(t_1) \dots d\mu^{eq}(t_k)$$

This proves (144.1) and hence Corollary 143.5.

We now use Corollary 143.5 to prove, finally, the following result.

Thm 146.1 Let $\phi : \mathbb{R}^k \to \mathbb{R}$ be a bded cont. function. Then

(146.2) $$\lim_{N\to\infty} \frac{1}{N^k} \int \phi(x_1,\ldots,x_k)\, R_k(x_1,\ldots,x_k)\, dx_1 \ldots dx_k.$$

$$= \int \phi(x_1,\ldots,x_k)\, q\mu^{eq}(t_1) \ldots q\mu^{eq}(t_k)$$

where R_k is the k^{th} correlation function. In particular for $k = 1$ we have for any bded, cont $\phi : \mathbb{R} \to \mathbb{R}$,

(146.3) $$\lim_{N\to\infty} \int \frac{\phi(x)}{N} R_1(x)\, dx = \int \phi(x)\, q\mu^{eq}(x)$$

which is (129.4), as desired.

In order to prove Thm 146.1, note that

$$\mathbb{E}xp_N \Big(\sum_{i_1,\ldots,i_k} \phi(x_{i_1},\ldots,x_{i_k}) \Big) = \mathbb{E}xp_N \Big(\sum_{\substack{i_1,\ldots,i_k \\ \text{distinct}}} \phi(x_{i_1},\ldots,x_{i_k}) \Big)$$

$$+ \mathbb{E}xp_N \Big(\sum_{\substack{i_j = i_e \text{ for} \\ \text{some } j \neq e}} \phi(x_{i_1},\ldots,x_{i_k}) \Big)$$

The # of terms in $\sum_{i_1,\ldots,i_k}$ is N^k and the # of terms in $\left(\sum_{\substack{i_1,\dots,i_k \\ \text{distinct}}}\right)$ is $N(N-1)\cdots(N-k+1)$. Hence the #

of terms in $\left(\sum_{\substack{i_j=i_\ell \\ \text{for some } j\neq \ell}}\right)$ is $O(N^{k-1})$. As

ϕ is bdd, we see that

$$\frac{1}{N^k}\,\mathbb{E}xp_N\Big(\sum_{\substack{i_j=i_\ell \text{ for some} \\ j\neq\ell}} \phi(x_{i_1},\dots,x_{i_k})\Big) = \frac{O(N^{k-1})}{N^k} \to 0$$

as $N\to\infty$. On the other hand, by symmetry,

$$\mathbb{E}xp_N\Big(\sum_{\substack{i_1,\dots,i_k \\ \text{distinct}}} \phi(x_{i_1},\dots,x_{i_k})\Big)$$

$$= \frac{N!}{(N-k)!}\int \phi(x_1,\dots,x_k)\, P_N(x)\, d^N x$$

$$= \int \phi(x_1,\dots,x_k)\, R_N(x_1,\dots,x_k)\, d^k x$$

and (146.2) follows immediately $\square$.

Remark:

As we have shown Th^m 146.1 follows directly

from Corollary 143.5. This corollary can be proved directly by applying the proof of Lemma (136.1) to $\frac{1}{N^k} \mathbb{E}xp_N \left(\sum \Phi(x_{i_1}, \dots, x_{i_k}) \right)$ rather than to $\frac{1}{N} \log \mathbb{E}xp_N \left(e^{N^{-k+1} \sum \Phi(x_{i_1}, \dots, x_{i_k})} \right)$.

(Exercise). We proceeded as above because Lemma 136.1 is of independent interest, and is needed to prove the following basic result (we will not prove it : see §6.5 in ref (2)):

Let $\pi_N(x) = x^N + \dots$, $N \geq 0$, be the monic orthogonal polynomials associated with the weight $e^{-NV(x)}dx$, and let $x^\# = x_1^\#, \dots, x_N^\#$ be the roots of $\pi_N(x)$ (the $x_i^\#$'s are real: why?)

Let

$$\mu_N^\# = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i^\#}$$

be the counting measure for the $x_i^\#$'s. Then

(148.1) $$\mu_N^\# \to \mu^{eq}$$

as $N \to \infty$.

Remark:

Here is 4th interpretation of μ^{eq} in addition to

- $E^V = H(\mu^{eq})$
- $\frac{1}{N} R_1(x)\,dx \to \mu^{eq}$, $N \to \infty$
- $\mu_N^\# \to \mu^{eq}$, $N \to \infty$

In (143.1) we introduced

$$d_N^V = \frac{1}{N(N-1)} \inf_{x \in \mathbb{R}^N} K_N(x) \tag{149.1}$$

The minimum in (149.1) is attained and any minimizer $x^0 = (x_1^0, \dots, x_N^0)$ is called a Fekete set.

Let

$$\mu_N^0 = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^0}$$

be the normalized measure for any Fekete set. Then (see Ref(2) §6.3)

- $\mu_N^0 \to \mu^{eq}$, $N \to \infty$.

Thus we see that μ^{eq} is the answer to many diff. analy. problems!

We now obtain Euler-Lagrange-type variational equations for the constrained variational problem for μ^{eq}

(150.1) $$E^V = \inf_{\mu \in M_1(\mathbb{R})} H(\mu)$$

$$= \inf_{\mu \in M_1(\mathbb{R})} \left[\iint \log|t-s|^{-1} d\mu(t)\, d\mu(s) + \int V(s)\, d\mu(s) \right]$$

$$= H(\mu^{eq}).$$

We will then use these variational equations to compute μ^{eq} for certain V's.

Recall that the (essential) <u>support</u> Σ_e $= supp(\mu)$ of a measure μ on Borel sets in $\mathbb{R}$ is the complement of the largest open set O_e for which $\mu_e(O) = 0$ (why does a largest such set $\exists$?). Recall also that Σ_e can be characterized by the condition

$$\Sigma = \{ x \in \mathbb{R} : \mu(x-\varepsilon, x+\varepsilon) > 0 \ \forall \varepsilon > 0 \}$$

Remark: If $\mu \in M_1(\mathbb{R})$ has compact (essential) support, it follows that, for fixed x, $\log|x-y|^{-1}$ is bded below on $\Sigma = \operatorname{supp}\mu$; hence, $\int \log|x-y|^{-1}\, d\mu(y)$ is well-defined for all x as an element of $(-\infty, \infty]$.

Th^m 151.1 (Variational equations: Weak form)

The eqm. meas μ^{eq} in (150.1) satisfies the following conditions:

There $\exists$ a real constant l such that

(i) $$\int \left[2 \int \log|x-y|^{-1}\, d\mu^{eq}(y) + V(x) \right] d\tilde{\mu}(x) \geq l$$

$\forall\, \tilde{\mu} \in M_1(\mathbb{R})$ of compact support with $H(\tilde{\mu}) < \infty$

(ii) $$2 \int \log|x-y|^{-1}\, d\mu^{eq}(y) + V(x) = l, \quad \mu^{eq}\text{-almost everywhere.}$$

Conversely, if $\mu \in M_1(\mathbb{R})$ has compact support, satisfies conditions (i) and (ii) above, and $H(\mu) < \infty$, then $\mu = \mu^{eq}$.