Lecture 11

We have shown the following: Let $V(x)$ be C^2 and satisfy $V(x)/\log(x^2+1) \to +\infty$ as $|x| \to \infty$.

Let $\Sigma = \bigcup_{i=1}^{k} (a_i, b_i)$

$a_1 \quad b_1 \qquad a_2 \quad b_2 \qquad - - - \qquad a_k \quad b_k$

be a disjoint union of intervals in $\mathbb{R}$, and set

$$G(z) = \frac{(q(z))^{\frac{1}{2}}}{2\pi i} \int_\Sigma \frac{\frac{i}{\pi} V'(s)}{(q(s))_+^{\frac{1}{2}}} \frac{ds}{s-z}$$

where

$$q(z) = \prod_{i=1}^{k} (z-a_i)(z-b_i)$$

and $(q(z))^{\frac{1}{2}}$ is analytic in $\mathbb{C} \setminus \Sigma$ normalized st $(q(z))^{\frac{1}{2}} \sim +z^k$ as $z \to \infty$.

let $\psi(x) \equiv \operatorname{Re} G_+(x)$, $x \in \mathbb{R}$. (As $V \in C^2$, $\psi(x) \in C_c$)

Suppose that the following are true

(169.1) $$\int_\Sigma \frac{V'(s)}{(q(s))_+^{\frac{1}{2}}} s^j ds = 0, \quad \text{for } j = 0, \dots, k-1$$

(169.2) $$\frac{i}{2\pi} \int_\Sigma \frac{V'(s)}{(q(s))_+^{\frac{1}{2}}} s^k ds = 1$$

(169.3) $$\int_{b_i}^{a_{i+1}} \left(H\psi(y) - \frac{V'(y)}{2\pi} \right) dy = 0, \quad i = 1, \dots, k-1.$$

where $H\psi(y) = \frac{1}{\pi} ⨍_\Sigma \frac{\psi(s)}{y-s} dy$

~~[Check $\psi(x) = \mathrm{Re}\, G_+(x)$]~~

(170.1) $$\psi(x) \geq 0 \quad \forall x \quad , \quad \{\psi(x) > 0\} = \Sigma$$

(170.2) (a) $$\int_{b_i}^{x} \left(H\psi(y) - \frac{V'(y)}{2\pi} \right) dy \leq 0 \quad , \quad b_i \leq x \leq a_{i+1} \, , \; i = 1, \dots, k-1$$

(b) $$\int_{x}^{a_1} \left(H\psi(y) - \frac{V'(y)}{2\pi} \right) dy \geq 0 \quad , \quad x \leq a_1$$

(c) $$\int_{b_k}^{x} \left(H\psi(y) - \frac{V'(y)}{2\pi} \right) dy \leq 0 \quad , \quad x > b_k$$

Then

$$d\mu^{eq}(x) = \psi(x)\, dx$$

By way of example, we consider

(170.3) $$V(x) = t\, x^{2m} \quad , t > 0 \, , \quad m \geq 1$$

We make the ansatz that $k = 1$, so $\Sigma = (a_1, b_1)$ for some $a_1 < b_1$. Then $q(z) = (z - a_1)(z - b_1)$ and

(170.4) $$G(z) = \frac{\left((z - a_1)(z - b_1)\right)^{\frac{1}{2}}}{2\pi i} \int_{a_1}^{b_1} \frac{\frac{2mit}{\pi} s^{2m-1}}{\left((s - a_1)(s - b_1)\right)_+^{\frac{1}{2}}} \frac{ds}{s - z}$$

and the moment condition (169.1) becomes

(170.5) $$\int_{a_1}^{b_1} \frac{s^{2m-1}}{\left((s - a_1)(s - b_1)\right)_+^{\frac{1}{2}}} ds = 0$$

which can clearly be satisfied by taking $(a_1, b_1) = (-a, a)$ for any $a > 0$. This is because

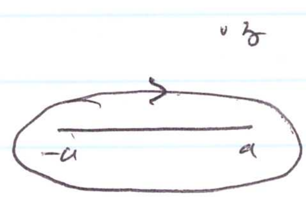
$$((s-a_1)(s-b_1))_+^{\frac{1}{2}} = ((s+a)(s-a))_+^{\frac{1}{2}} = i\sqrt{a^2-s^2}$$

for $-a < s < a$, and we use the convention that $\sqrt{\alpha}$ always denotes the pos. square root of a pos. # α, and s^{2m-1} is odd, while $\sqrt{a^2-s^2}$ is even, $-a < s < a$.

Now by analyticity, for $z \in \mathbb{C} \setminus \overline{\Sigma}$, $\quad \overline{-a \qquad a}$

$$G(z) = \frac{(z^2-a^2)^{\frac{1}{2}}}{4\pi i} \int_{\mathcal{C}} \frac{\frac{2mit}{\pi} s^{2m-1}}{(s^2-a^2)^{\frac{1}{2}}} \frac{ds}{s-z}$$

where $\mathcal{C}$ is a contour around $\overline{\Sigma}$, with z in its exterior



Thus by Cauchy, for $R > |z|$,

$$G(z) = \frac{mit}{\pi} z^{2m-1} - \frac{mit}{\pi} \frac{(z^2-a^2)^{\frac{1}{2}}}{2\pi i} \oint_{|s|=R} \frac{s^{2m-1}}{(s^2-a^2)^{\frac{1}{2}}} \frac{ds}{s-z}$$

where $\oint_{|s|=R}$ denote counter-clockwise integration on the circle $|s| = R$.

Letting $R \to \infty$, and doing a residue calculation (exercise), we find

(172.1) $$G(z) = \frac{mit}{\pi}\left(z^{2m-1} - (z^2-a^2)^{\frac{1}{2}} h_1(z)\right)$$

where

(172.2) $$h_1(z) = z^{2m-2} + \sum_{j=1}^{m-1} z^{2m-2-2j} a^{2j} \prod_{l=1}^{j} \frac{(2l-1)}{2l}$$

If $m=1$, $h_1(z) = 1$

We see that for $|x| \geq a$, $\psi(x) = \operatorname{Re} G_+(x) = 0$

and for $-a < x < a$

(172.3) $$\psi(x) = \frac{mt}{\pi} \sqrt{a^2-x^2}\, h_1(x) > 0$$

Thus (170.1) is satisfied

Now (169.2), fixes a,

(172.4) $$\frac{1}{2\pi}\int_{-a}^{a} \frac{2mt\, s^{2m}}{(a^2-s^2)^{\frac{1}{2}}}\, ds = 1.$$

Again the LHS can be evaluated by a residue calculation.

However observe that, from (170.4) and (172.1)

$$\frac{q_+^{\frac12}(0)}{2\pi i}\int_{-a}^{a}\frac{\frac{2mit}{\pi}\, s^{2m-2}}{(q(s))_+^{\frac12}}\,ds = G_+(0)$$

$$= \left(\frac{mit}{\pi}\right)\left(-q_+^{\frac12}(0)\right)h_1(0).$$

and so

$$\frac{1}{2\pi i}\int_{-a}^{a}\frac{s^{2m-2}}{(q(s))_+^{\frac12}}\,ds = -h_1(0)$$

or, using (172.2)

$$h_1(0) = a^{2(m-1)}\prod_{l=1}^{m-1}\frac{2l-1}{2l} = \frac{2}{2\pi}\int_{-a}^{a}\frac{s^{2m-2}}{\sqrt{a^2-s^2}}\,ds$$

Replacing $m-1$ by m in the above identity, we find that (172.4) reduces to

$$mt\, a^{2m}\prod_{l=1}^{m}\frac{2l-1}{2l} = 1.$$

Thus

(173.1) $$a = \left(mt\prod_{l=1}^{m}\frac{2l-1}{2l}\right)^{-1/2m}$$

As $k=1$, there are no gaps and so (169.2) and (170.2)(b) do not need to be checked. Now recall from (158.2), $G(z) = \frac{1}{i\pi}\int\frac{\psi(y)}{y-z}\,dy$, $z\in\mathbb{C}\setminus\Sigma$

and so for $x>a$ or $x<-a$,

$$G(x) = \frac{1}{i\pi} \int_{-a}^{a} \frac{\psi(y)}{y-x}\, dy = i\, H\psi(x)$$

In particular, for $x > a$

$$\int_a^x \left(H\psi(y) - \frac{2mty^{2m-1}}{2\pi} \right) dy$$

$$= \int_a^x \left(-i\, G(y) - \frac{2mty^{2m-1}}{2\pi} \right) dy.$$

$$\underset{(172.1)}{=} \int_a^x \left[\frac{mt}{\pi} \left(y^{2m-1} - \sqrt{y^2-a^2}\, h_1(y) \right) - \frac{mt}{\pi} y^{2m-1} \right] dy.$$

$$= -\frac{mt}{\pi} \int_a^x (y^2-a^2)^{\frac{1}{2}} h_1(y)\, dy < 0$$

so that (170.2)(a) is satisfied: The case (170.2)(c) is similar. Hence

(174.1) $$\psi(x)\, dx = \operatorname{Re} G_+(x)\, dx = \frac{mt}{\pi} \sqrt{a^2-x^2}\; h_1(x)\; \chi_{(-a,a)}(x)\, dx$$

is the equilibrium measure for $V = tx^{2m}$, $m \geq 1$,

where

$$a = \left(mt \prod_{l=1}^{m} \frac{2l-1}{2l} \right)^{-\frac{1}{2m}}$$

Note that if $m=1$, then $a = \left(\frac{t}{2}\right)^{-\frac{1}{2}} = \sqrt{\frac{2}{t}}$

(175.1) $$\psi(x)\,dx = \frac{t}{\pi}\sqrt{\frac{2}{t} - x^2}\;\chi_{\left(-\sqrt{\frac{2}{t}},\,\sqrt{\frac{2}{t}}\right)}(x)\,dx$$

as $h_1(x) = 1$. This is the celebrated Wigner semi-circle law. Note that

$$h_1(x) \geq h_1(0) > 0 \quad , \forall x$$

and so $\psi(x)\,dx$ always has a square root singularity at $\pm a$.

<u>Another example</u>: Consider $V(x) = x^4 - t x^2$, $t \geq 0$.

Show that if we assume $k=1$, then $\Sigma = (-a, a)$ gives rise to a solution of (169.1) and (169.2) if

$$a^2 = \frac{t + \sqrt{t^2 + 12}}{3}$$

In this case

$$G(z) = \frac{i}{2\pi}(3z^3 - 2tz) - \frac{i}{\pi}(z^2 - a^2)^{\frac{1}{2}}(a^2 + 2z^2 - t)$$

$$= \frac{i}{2\pi}(3z^3 - 2tz) - \frac{i}{\pi}(z^2 - a^2)^{\frac{1}{2}}\left(\frac{\sqrt{t^2+12} - 2t + 6z^2}{3}\right)$$

Hence

(176.1) $$\psi(x) = \operatorname{Re} G_+(x) = \frac{\sqrt{a^2-x^2}}{3\pi}\left(\sqrt{t^2+12}-2t\right) \times \left(\sqrt{t^2+12}-2t+6x^2\right)\chi_{(-a,a)}(x)$$

In particular we see that

$$\psi(0) = \frac{a}{3\pi}\left(\sqrt{t^2+12}-2t\right) \geq 0$$

$\iff t \leq 2$.

Thus $\mu^{ed}(x)$ cannot be given by one-interval solution $\Sigma = (a_1, b_1)$ if $t > 2$. If $t \leq 2$, then $\psi(x) \geq 0$ on $(-a, a)$. Also for $x > a$,

$$\int_a^x \left(H\psi(y) - \frac{4y^3-2ty^2}{2\pi}\right) dy$$

$$= \int_a^x \left(-iG(y) - \frac{4y^3-2ty^2}{2\pi}\right) dy$$

$$= -\int_a^x \frac{1}{\pi}(y^2-a^2)^{\frac{1}{2}} \frac{\left(\sqrt{t^2+12}-2t+6y^2\right)}{3} dy < 0$$

and similarly $\int_x^{-a} \left(H\psi(y) - \frac{4y^3-2ty^2}{2\pi}\right) dy > 0$ for $x < -a$.

Thus $\mu^{ed} = \psi(x)\,dx$ with $\psi(x)$ in (176.1), in the case $0 < t \leq 2$

However, as noted above, (176.1) cannot give μ^{eq} for $t>2$. <u>Exercise</u>: Show that μ^{eq} can be expressed in terms of a 2 interval solution

$$\Sigma = (-b,-a) \cup (a,b) \quad \text{for} \quad 0<a<b.$$

Note that for $t=2$, $\psi(0)=0$, but $\psi(x)>0$ for $0<|x|<a$. Thus a gap is opening up at $x=0$ as t crosses 2.

Universality

We have shown that in the case $\beta=2$ (unitary ensembles) that basic statistical quantities for the eigenvalues can be expressed in terms of the correlation kernel $K(x,y)$ where

(178.1) $$K(x,y) = \sum_{j=0}^{N-1} \phi_j(x)\, \phi_j(y)$$

and $$\phi_j(x) = \sqrt{w(x)}\, p_j(x) \quad , \quad j \geq 0$$

where $p_j(x) = \gamma_j x^j + \cdots$, $\gamma_j > 0$, $j \geq 0$, are the orthonormal polynomials wrt the weight $w(x)\,dx$ i.e.

$$\int p_j(x)\, p_k(x)\, w(x)\, dx = \delta_{jk} \quad , \quad j,k \geq 0.$$

In particular from (89.2), for $\Omega \subset \mathbb{R}$

(178.2) $$\text{Prob}\,(\text{no eigenvalues in } \Omega) = \det\left(1_{L^2(\mathbb{R})} - K\chi_\Omega\right)$$

and from (100.3) the k-point correl. func. is given by

$$R_k(x_1, \ldots, x_k) \equiv \det\left(K(x_i, x_j)\right)_{1 \leq i,j \leq k}$$

The basic fact is that as $N \to \infty$, these basic statistics behave <u>universally</u> on an appropriate scale. And as noted, this means that we must be able to evaluate the assymptotics (as $N \to \infty$) of the polynomials $p_j(x)$ that are the building blocks for the correlation kernel via (178.1).

In the Gaussian case where $V(x) = x^2$ (GUE) the associated polynomials $\int p_j(x)\, p_k(x)\, e^{-x^2} dx = \delta_{jk}$ can be expressed in terms of the well-known Hermite polynomials $H_j(x)$; one has the classical formulae (see [Szegő], Orthog. Polynomials, Chap V, Sect. 5.5)

(179.1) $$\int_{-\infty}^{\infty} e^{-x^2} H_e(x)\, H_m(x)\, dx = \pi^{\frac{1}{2}} 2^m m!\, \delta_{e,m}$$

where

(179.2) $$H_m(x) = 2^m x^m + \cdots, \quad m \geq 0$$

The polynomials can be written down explicitly (see [Szegő], eqn (5.5.4))

(179.3) $$H_m(x) = m! \sum_{\nu=0}^{[\frac{m}{2}]} \frac{(-1)^\nu}{\nu!} \frac{(2x)^{m-2\nu}}{(m-2\nu)!} = (2x)^m + \cdots$$

In terms of the scaled weight $e^{-Nx^2}dx$, the monic orthogonal polynomials $\pi_j(x) = x^j + \dots$, $p_j(x) = \gamma_j x^j + \dots$, $\int e^{-nx^2}\pi_j(x)\,\pi_k(x)\,dx = 0$ for $j \neq k$, we have

(180.1) $$\pi_m(x) = \frac{1}{(4N)^{m/2}} H_m(\sqrt{N}\,x) = \frac{1}{(4N)^{m/2}}\left(2^m(\sqrt{N}x)^m + \dots\right) = x^m + \dots$$

and

$$\int e^{-Nx^2}\pi_\ell(x)\,\pi_m(x)\,dx$$

$$= \int e^{-Nx^2}\frac{1}{(4N)^{\frac{\ell+m}{2}}} H_\ell(\sqrt{N}x)\,H_m(\sqrt{N}x)\,dx$$

$$= \frac{1}{(4N)^{\frac{\ell+m}{2}}}\frac{1}{\sqrt{N}}\int e^{-u^2}H_\ell(u)\,H_m(u)\,du$$

$$= \frac{1}{(4N)^{\frac{\ell+m}{2}}}\frac{1}{\sqrt{N}}\,\pi^{\frac{1}{2}}\,2^m\,m!\,\delta_{\ell,m}$$

$$= \frac{\pi^{1/2}\,m!}{2^m\,N^{m+\frac{1}{2}}}\,\delta_{\ell,m}$$

from which we see, in particular, that

(181.1) $$\gamma_m^{-2} = \int \pi_m^2(x)\, e^{-nx^2}\, dx$$

$$= \frac{\pi^{1/2}\, m!}{2^m N^{m+\frac{1}{2}}}.$$

i.e.

(181.2) $$\gamma_m^2 = \frac{2^m N^{m+\frac{1}{2}}}{\sqrt{\pi}\, m!}$$

Now it is possible to evaluate the asymptotics of the H_m's, and hence the asymptotics for the correlation kernel $K(x,y)$ in the case of GUE, because of an exceptional circumstance viz the H_m's can be expressed in terms of a contour integral: indeed we have (see Szegő V Sec 5.5)

(181.3) $$H_m(x) = \frac{m!}{2\pi i} \int_{\mathcal{C}} w^{-m-1} e^{(2xw - w^2)}\, dw$$

where the contour $\mathcal{C}$ is counter clockwise around the origin.

The asymptotics of $H_m(x)$ as $m \to \infty$, then follows vi the standard way by applying the classical steepest descent method to (181.3). For example, using this method, one finds that for $\varepsilon > 0$

$$x = (2n+1)^{\frac{1}{2}} \cos\phi , \quad \varepsilon \le \phi \le \pi - \varepsilon$$

(182.1)
$$H_n(x) = e^{x^2/2}\, 2^{\frac{n}{2}+\frac{1}{4}} (n!)^{\frac{1}{2}} \frac{1}{(\pi n)^{\frac{1}{4}}} (\sin\phi)^{-\frac{1}{2}} \times \left\{ \sin\left[\left(\frac{n}{2}+\frac{1}{4}\right)(\sin 2\phi - 2\phi) + \frac{3\pi}{4}\right] + O\left(\frac{1}{n}\right) \right\}$$

as $n \to \infty$ (see [Szegő] §8.28 p201)

(**) Exercise! derive (182.1) from (181.3).

If one inserts (182.1) into (178.2) with

$$\Omega_N = \left(x_0 - \frac{\theta}{\psi(x_0) N} , \; x_0 + \frac{\theta}{\psi(x_0) N}\right) , \quad \theta > 0$$

where x_0 is any point in the support of $d\mu^{eq}$ $= \psi(x)\,dx = \frac{1}{\pi}\sqrt{2 - x^2}\, \chi_{(-\sqrt{2},\sqrt{2})}(x)\,dx$ (cf (175.1), the eqm. meas for GUE $e^{-Nx^2}dx$) i.e. $|x_0| < \sqrt{2}$

one finds, as we will see that

(183.1) $$\text{Prob}\left(\text{no evs's in } \left(x_0 - \frac{\theta}{\psi(x_0)N}, x_0 + \frac{\theta}{\psi(x_0)N}\right)\right)$$

$$\underset{N\to\infty}{\longrightarrow} \det(1 - S)_{L^2(-\theta,\theta)}$$

where S acts on $L^2(-\theta,\theta)$ with kernel

$$S(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$$

Universality means that we will obtain precisely the same formula $\det(1-S)_{L^2(-\theta,\theta)}$ for the scaled gap probability for all reasonable weights $e^{-NV(x)}dx$. But in order to show this we need to control the asymptotics of the polynomials $\{p_j(x)\}$ orthonormal wrt $e^{-NV(x)}dx$. In general, however, such polynomials do not have an integral representation such as (181.3). One needs to find a general method to evaluate such polynomial asymptotics and this is where the Riemann-Hilbert Problem (RHP) comes in: every ortho. polynomial has a representation in terms of a RHP, and the asymptotics of the OP's can be extracted from the RHP by using a <u>non-linear</u> steepest-descent method for RHP's that is as effective and explicit as the classical steepest descent method for integral representations.