Random Matrix Theory (RMT) I

Fall 2013

Prerequisites : Linear Algebra, Complex Variables, Functional Analysis, Probability Theory

References :

1. M. L. Mehta Random Matrices 3rd Edit.
2. P. Deift Orthogonal Polynomials and Random Matrices : A Riemann - Hilbert Approach
3. P. Deift and D. Gioev Random Matrix Theory : Invariant Ensembles and Universality
4. P. J. Forrester, Log - gases and random matrices
5. G. Akemann, J. Baik and P. Di Francesco, The Oxford Handbook of Random Matrix Theory
6. T. Tao, Topics in Random Matrix Theory

(7) G. Anderson, A. Guionnet & O. Zeitouni, An Introduction to Random Matrices

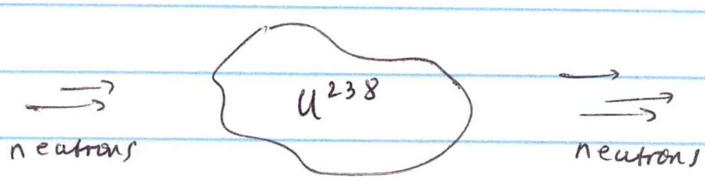
(8) G. Blower, Random matrices: High Dimensional Phenomena

(9) P. Deift, Universality for physical & mathematical systems

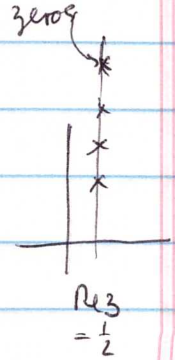
(10) J. Baik, P. Deift and T. Suidan, Combinatorics & RMT

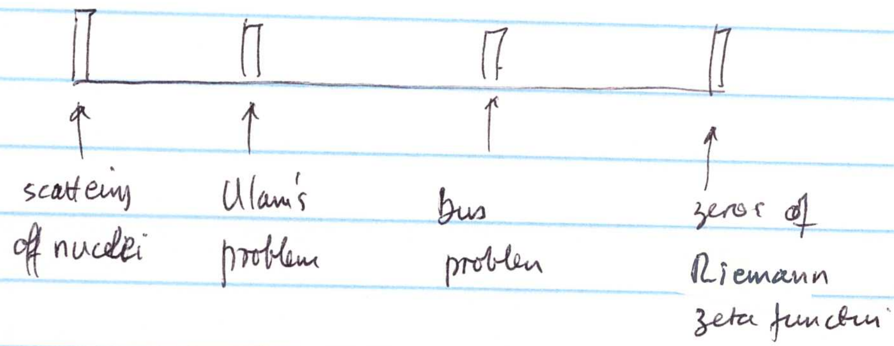
A random matrix $M = \{M_{ij}\}$ is, of course, a matrix whose entries are chosen randomly. There are many different kinds of random matrix ensembles, comprising Hermitian matrices, unitary matrices, orthogonal matrices, sample covariance matrices, ... (see [Mehta], [Ake Baik Di Fr])

What is remarkable about random matrix theory, and what accounts for the great interest RMT has generated in the physics, mathematics and engineering communities, is that the eigenvalues of random matrices provide a model for an astonishing variety of problems across the scientific spectrum. In particular, sitting as bookends on the scientific shelf, we now know that the scattering resonances of neutrons off heavy nuclei



at the one end, and the zeros of the Riemann zeta function on the critical line, at the other end, are described, on the appropriate scale by the eigenvalues of a (large) random matrix. And in between these bookends





we find combinatorial problems such as Ulam's longest increasing subsequence problem, and transportation problems, such as the so-called "Bus Problem" in Cuernavaca, Mexico, together with a myriad of other problems from a variety of scientific areas, all of which are described by one aspect or the other random matrix theory.

As a subject, RMT has two aspects.

(1) <u>The intrinsic theory</u>
Here one studies RMT per se, independent of any applications. This activity mirrors in

spirit Watson's classic text "Bessel Functions", for example, where the basic properties of Bessel functions are laid out

(2) <u>applications</u>

Here one considers the application of RMT to concrete problems in science; (2) of course draws on (1).

This is a two-semester course: in the first semester we will concentrate on (1), and in the second semester we will concentrate on (2).

As there is a lot of material to cover, my hope is that we are able to understand in detail at least one of the applications of RMT to a concrete problem e.g Ulam's longest increasing subsequence problem. In order to do this, we will not give full proofs of all the technical results that we will need: fortunately there are many texts available where you can find all the details.

I want to spend the rest of this lecture just sketching some of the applications of RMT. (see P. Deift (4) for more details & ref's)

(i) <u>Zeros of the Riemann-zeta function on the critical line</u> (one of the bookends mentioned above)

Assuming the Riemann-Hypothesis the zeros of $\zeta(s)$ are given by $s_j = \frac{1}{2} + i\gamma_j$, $\gamma_j > 0$ (and also of course $\bar{s}_j$). In the early 70's, H. Montgomery, began investigating the distribution of

γ_j's as $j \to \infty$. First he rescaled the γ_j's

$$\gamma_j \to \tilde{\gamma}_j = \frac{\gamma_j \log \gamma_j}{2\pi} \qquad j \geq 1$$

to have mean spacing 1 as $T \to \infty$ i.e.

(7.1) $$\lim_{T \to \infty} \frac{\#\{j \geq 1 : \tilde{\gamma}_j \leq T\}}{T} = 1$$

As we will see, analogous scalings are essential in RMT. For any $a < b$, he then computed the two-point correlation function for the $\tilde{\gamma}_j$'s

$$\#\{\text{ordered pairs } (j_1, j_2),\ j_1 \neq j_2 : 1 \leq j_1, j_2 \leq N,\ \tilde{\gamma}_{j_1} - \tilde{\gamma}_{j_2} \in (a,b)\}$$

and showed, modulo certain technical restrictions, that

(7.2) $$R(a,b) \equiv \lim_{N \to \infty} \frac{1}{N} \#\{\text{ordered pairs } (j_1, j_2),\ j_1 \neq j_2 : 1 \leq j_1, j_2 \leq N,\ \tilde{\gamma}_{j_1} - \tilde{\gamma}_{j_2} \in (a,b)\}$$

exists and is given by a certain explicit formula.

The following story has been told many times.

Soon after completing his work on the scaling limit (7.2), Montgomery was visiting the Institute for Advanced Study in Princeton and it was suggested that he show his result to Freeman Dyson. This is what happened: Before Montgomery could describe his hard won result to Dyson, Dyson took out a pen, wrote down a formula, and asked Montgomery, "And did you get this?"

(8.1) $$R(a,b) = \int_a^b \left[1 - \left(\frac{\sin \pi r}{\pi r}\right)^2\right] dr$$

Montgomery was stunned: This was exactly the formula that he had worked so hard to obtain. Dyson explained; "If the zeros of the zeta function behaved like the eigenvalues of a random matrix from the Gaussian Unitary Ensemble (GUE) (see below!)

then (8.1) would be exactly the formula for the two-point correlation function of the eigenvalues!"

This relationship between the zero's of the zeta function and RMT first discovered by Montgomery has been taken up with great virtuosity by various authors over the years (Rudnick-Sarnak, Katz-Sarnak, Keating-Snaith, ...). Spectacular numerical verification of this relationship due to Andrew Odlyzko can be seen, for example, in Mehta's book.

(ii) <u>Ulam's longest increasing subsequence problem</u>

Consider the space S_N of permutations π of the numbers $1, 2, \ldots, N$. We say that if

$$i_1 < i_2 < \ldots < i_k$$

and

$$\pi(i_1) < \pi(i_2) < \ldots < \pi(i_k)$$

then $\pi(i_1), \dots, \pi(i_k)$ is an increasing subsequence of π of length k. Let $l_N(\pi)$ be the maximal length of all increasing subsequences of π.

For example if $\pi = 341562$ i.e. (N = 6 and $\pi(1) = 3, \pi(2) = 4, \dots$ $\pi(6) = 2$), then 34 is an inc. subseq. of length 2, 356 is an inc. subseq. of length 3, and 3456 is an increasing subsequence of maximal length, $l_6(\pi) = 4$.

Equip S_N with uniform measure: thus $\text{Prob}(\{\pi\}) = \frac{1}{N!}$ for all $\pi \in S_N$. Consider the distribution function for l_N:

$$p_{n,N} = \text{Prob}(\pi : l_N(\pi) \le n)$$

Then as $n, N \to \infty$, one can show (Baik, D, Johansson, 1999) (see also Ref (10))

(10.1) $$\lim_{N\to\infty} \text{Prob}\left(\pi : \frac{l_N(\pi) - 2\sqrt{N}}{N^{1/6}} \le t\right) = F(t)$$

where

(10.2) $$F(t) = e^{-\int_t^\infty (x-t)\, u^2(x)\, dx}$$

where $u(x)$ is the unique solution of the Painlevé II equation (more later)

$$u''(x) = x\,u(x) + 2u^3(x)$$

normalized so that

$$u(x) \sim Ai(x) \quad \text{as } x \to +\infty$$

Here $Ai(x)$ is the classical Airy function, $Ai''(x) = x\,Ai(x)$.

The surprising fact is the following: $F(t)$ is precisely the distribution function for the largest eigenvalue of a GUE random matrix, in the so called edge-scaling limit as the size N of the matrices goes to infinity. In other words, l_N behaves like the largest eigenvalue of a (large) GUE matrix!

There are many ways to restate this result, one of which is known as "patience sorting".

Consider a pack of N cards, numbered $1, 2, \ldots, N$, and play the following game. The deck is shuffled and the first card is placed face up on the table in front of the dealer. If the next card is smaller than the card on the table, it is placed face up on top of the card: if it is bigger, the card is placed face up to the right of the first card, making a new pile. If the third card in the pile is smaller than one of the cards on the table, it is placed on top of that card: if it is smaller than both cards, it is placed face up to the right of the pile(s), making a new pile. One continues in this fashion until all the cards are played out: let q_N be the # of piles obtained. Clearly q_N depends on the shuffle,

$q_N = q_N(\pi)$, $\pi \in S_N$.

For example if $N = 6$ and $\pi = 341562$, where 3 is the top card etc, Then the game goes as follows:

$$3 \qquad 3\;4 \qquad \begin{array}{c}1\\3\end{array}\;4 \qquad \begin{array}{c}1\\3\end{array}\;4\;5 \qquad \begin{array}{c}1\\3\end{array}\;4\;5\;6 \qquad \begin{array}{c}1\\3\end{array}\begin{array}{c}2\\4\end{array}\;5\;6$$

so $q_6(\pi) = 4$. Note that $q_6(\pi) = \ell_6(\pi)$.

This is no accident! Prove the following:

Exercise: For any $\pi \in S_N$

$$\ell_N(\pi) = q_N(\pi).$$

So we may restate (10.1) (10.2) as follows: the # of piles in patience sorting behaves like the largest eigenvalue of a (large) GUE matrix.

In addition to convergence in distribution (10.1) (10.2), one can also prove convergence of the moments, and one finds in particular that

(14.1) $$\lim_{N\to\infty} \mathbb{E}xp\left(\frac{q_N - 2N^{\frac{1}{2}}}{N^{1/6}}\right) = \int t\, dF(t)$$

Numerical computation for the RHS shows that as $N \to \infty$

(4.2) $$\mathbb{E}xp(q_N) \sim 2N^{\frac{1}{2}} - 1.7711\, N^{\frac{1}{6}}$$

Thus if you are betting against your friends in a bar one evening how big a table you need to play the game with a standard pack $(N=52)$, then you know how to bet!

$$\mathbb{E}xp(q_{52}) \sim 2\sqrt{52} - 1.7711\, N^{\frac{1}{6}}$$

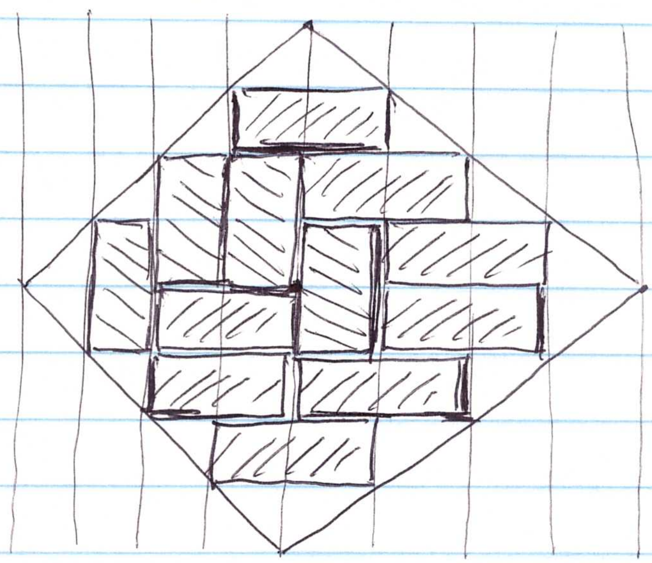
$$\sim 11 \text{ units}$$

(iii) <u>Tiling an Aztec diamond</u>

Consider tilings $\{T\}$ of the tilted square

$$SQ_n = \{(x,y) : |x| + |y| \leq n+1\}$$

in $\mathbb{R}^2$ by horizontal and vertical dominoes of length 2 and width 1. For example for $n=2$ we have the tiling T



$n=3$

Each tiling must lie strictly within SQ_n. Tilings T are called Aztec diamonds because the bdry of T in $\{(x,y) : y>0\}$ has the shape of a Mexican pyramid. It is a non-trivial result of Elkies et al, that there are precisely $2^{\frac{n(n+1)}{2}}$ (Exercise: Verify this result for $n=2$).

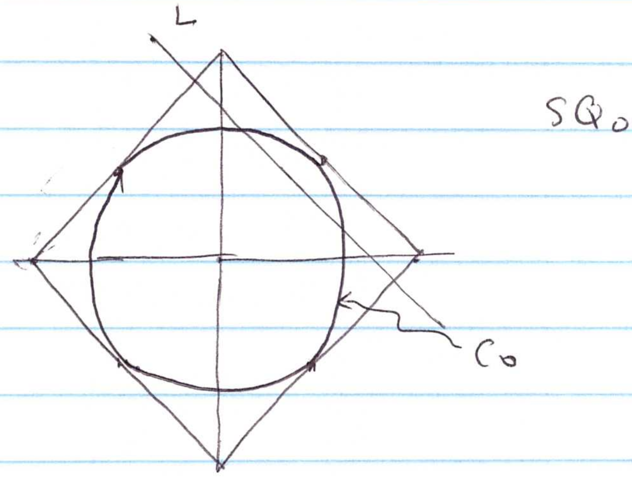
Ques: What does a typical tiling look like as $n \to \infty$?

After rescaling by $n+1$, Jockush et al in 1998 considered the tiling problem with dominos of size $\frac{2}{n+1} \times \frac{1}{n+1}$ in the tilted square $SQ_0 = \{ |u| + |v| \leq 1 \}$. As $n \to \infty$, they found that the inscribed circle $C_0 = \{(u,v) : u^2 + v^2 = \frac{1}{2}\}$, which they called the arctic circle, plays a remarkable role. In the 4 regions of SQ_0 outside C_0, which they call polar regions and label N, E, S and W clockwise from the top, the typical tiling is frozen, with all the dominoes in N and S horizontal and all the dominos in E and W vertical. In the region inside C_0, which they call the temperate zone, the tiling is random (see, e.g.,

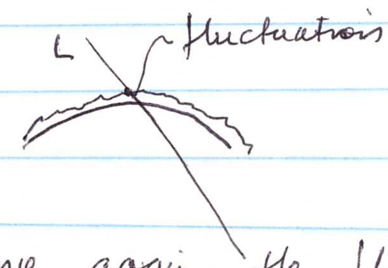
http://www.math.wisc.edu/~propp/tiling

where a tiling with $n=50$ is displayed).

But more is true. In 2001, 2002 Johansson considered fluctuations of the temperate zone about C_0. Visually



For any line L parallel to a side of SQ_0, he analyzed the fluctuations of the temperate zone along L



and found that, once again the fluctuations behaved like the largest eigenvalue of a large GUE matrix!

We hope to analyze Ulam's problem and/or the Aztec diamond problem in detail in the spring semester.

Additional Reference: For very recent developments in RMT, see the review of L. Erdős on the ArXiv: 1004.0861

(11) Universality of Wigner random matrices: a Survey of Recent Results

And even more recently

(12) L. Erdős and H. T. Yau, A Dynamical Approach to Random Matrix Theory