Lecture 4 (i) Harmonic oscillator $M = \mathbb{R}^2$, $H = \frac{1}{2}(p^2 + w^2 q^2) \equiv \phi_1$ where $w \neq 0$ (56)

Claim: H is integrable on the (invariant) domain $D = M/\{0\} = \mathbb{R}^2/\{0\}$.

The system is integrable as ϕ_1 is conserved and $n = 1$

and $dH = p\,dp + w^2 q\,dq \neq 0$ (except for $(p,q) = 0$) (clearly all non-degenerate Hamiltonians on 2-dimensional manifold are integrable!).

$N_0 = \{(q,p) : H = \phi_1 = C > 0\} = \{(q,p) : p^2 + w^2 q^2 = 2C\}$,

which is clearly a torus. The equations of motion are

$$\dot{q} = H_p = p \quad , \quad \dot{p} = -H_q = -w^2 q$$

with solution

$$q = \frac{\sqrt{2C}}{w} \sin(wt + \alpha) \ , \quad p = \sqrt{2C} \cos(wt + \alpha)$$

Note that $p^2 + w^2 q^2 = 2C$. Note that the set $D = \mathbb{R}^2/\{0\}$ is invariant under the flow generated by H.

The map ψ in the Theorem is constructed as follows. We can take $D_1 = \mathbb{R}^+$. Then

$$\mathbb{R}^+ \times \mathbb{T}^1 \ni (y, x) \mapsto \psi(y,x) = (q(y,x), p(y,x))$$
$$\equiv \left(\sqrt{\frac{y}{\pi w}} \sin 2\pi x , \sqrt{\frac{wy}{\pi}} \cos 2\pi x\right)$$

and $\psi^{-1} : \mathbb{R}^2 \to \mathbb{R}^+ \times \mathbb{T}^1$ takes

$$(q, p) \to (y, x) = \left(\frac{\pi}{w}(w^2 q^2 + p^2), \frac{1}{2\pi} \sin^{-1}\left(q \sqrt{\frac{\pi w}{y}}\right)\right)$$

We have $H \circ \psi(y,x) = \frac{1}{2}\left(\left(\sqrt{\frac{wy}{\pi}} \cos 2\pi x\right)^2 + w^2 \left(\sqrt{\frac{y}{\pi w}} \sin 2\pi x\right)^2\right) = \frac{w}{2\pi} y$

and

$$\psi^{*}(dq \wedge dp) = \left(\frac{1}{2}\frac{1}{\sqrt{\pi w y}}\sin 2\pi x\, dy + \sqrt{\frac{y}{\pi w}}\cos 2\pi x\ 2\pi\, dx\right)\wedge$$

$$\left(\frac{1}{2}\sqrt{\frac{w}{\pi y}}\cos 2\pi x\, dy - \sqrt{\frac{wy}{\pi}}\sin 2\pi x\ 2\pi\, dx\right)$$

$$= dx \wedge dy$$

In the (x, y) variables the flow becomes

$$\dot{x} = \frac{\partial}{\partial y} H\circ\psi = \frac{w}{2\pi} \quad , \quad \dot{y} = -\frac{\partial}{\partial x} H\circ\psi = 0$$

so that $x(t) = \frac{wt}{2\pi} + x_0$, $y(t) = y_0$, which

implies

$$q(t) = \sqrt{\frac{y_0}{\pi w}}\sin(wt + 2\pi x_0)$$

$$p(t) = \sqrt{\frac{w y_0}{\pi}}\cos(wt + 2\pi x_0)$$

as it should.

Exercise Compute ψ^{-1} and show directly that it is symplectic

(ii) Simple pendulum

Here $M^2 = (\mathbb{T} \times \mathbb{R}$, $\omega = dq \wedge dp$) and

$H = \frac{1}{2}p^2 + 1 - \cos 2\pi q$, which gives rise to the motion

$$\dot{q} = H_p = p \quad , \quad \dot{p} = -H_q = -2\pi \sin 2\pi q$$

or

$$\ddot{q} + 2\pi \sin 2\pi q = 0$$

Note that for x small $H \sim \frac{1}{2}y^2 + \frac{1}{2}(2\pi)^2 x^2$, so for x small the pendulum acts like a simple harmonic oscillator.

Exercises

(a) The motion of the pendulum depends on the value of $H = c > 0$.

Show that there are 3 different cases,

$$c < 2, \quad c = 2, \quad c > 2.$$

If $c < 2$ the pendulum oscillates back & forth with $|2\pi q(t)| < \pi$. If $c > 2$, the pendulum rotates "over the top". If $c = 2$, the pendulum moves from $2\pi q = -\pi$ to $2\pi q = +\pi$ as t runs from $-\infty$ to $+\infty$: This case is the so-called separatrix for the system

(b) Describe $N_0 = \{(x, y) : H(x, y) = c\}$ in the above three cases and draw a picture of M^2 foliated by the invariant sets $N_0 = N_0(c)$ for all values of $c > 0$.

(c) Construct the maps ψ and ψ^{-1} in the L-A-Jost theorem (Th^m 48.1) in this case.

We now begin studying the Toda lattice. As noted in lecture 1, the Toda lattice was introduced by M. Toda in 1967 and describes the motion of N particles x_i, $i=1,\dots,N$, on the $\mathbb{R}$ generated by the Hamiltonian

(59.1) $$H_T(x,y) = \frac{1}{2}\sum_{i=1}^{N} y_i^2 + \sum_{i=1}^{N-1} e^{x_i - x_{i+1}}, \quad i=1,\dots,N.$$

on the symplectic manifold $M^{2n} = \left(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{N} dx_i \wedge dy_i\right)$.

The Toda equations

(59.2) $$\begin{cases} \dot{x}_i = \dfrac{\partial H_T}{\partial y_i} = y_i \\ \dot{y}_i = -\dfrac{\partial H_T}{\partial x_i} = -e^{(x_i - x_{i+1})} + e^{(x_{i-1} - x_i)} \\ \qquad 2 \le i \le N-1 \\ \dot{y}_1 = -e^{(x_1 - x_2)}, \quad \dot{y}_N = e^{(x_{N-1} - x_N)} \end{cases}$$

Following Flaschka (alternatively, Manakov), set

(59.2) $$\begin{aligned} X_{ii} &= -y_i/2, \quad i=1,\dots,N \\ X_{i\,i+1} &= X_{i+1,i} = \frac{1}{2} e^{\frac{1}{2}(x_i - x_{i+1})}, \quad 1 \le i \le N-1, \end{aligned}$$

and consider the real, symmetric matrix

$$X = \begin{pmatrix} X_{11} & X_{12} & 0 & & \\ X_{21} & X_{22} & X_{23} & & 0 \\ & X_{32} & \ddots & \ddots & \\ & & \ddots & \ddots & X_{N-1\,N} \\ 0 & & & X_{N\,N-1} & X_{NN} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{y_1}{2} & \frac{1}{2}e^{\frac{1}{2}(x_1-x_2)} & 0 & 0 & & \\ \frac{1}{2}e^{\frac{1}{2}(x_1-x_2)} & -\frac{y_2}{2} & \frac{1}{2}e^{\frac{1}{2}(x_2-x_3)} & & 0 & \\ & \frac{1}{2}e^{\frac{1}{2}(x_2-x_3)} & -y_3/2 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & \frac{1}{2}e^{\frac{1}{2}(x_{N-1}-x_N)} \\ & & & & \frac{1}{2}e^{\frac{1}{2}(x_{N-1}-x_N)} & -y_N/2 \end{pmatrix}$$

Now for $2 \le i \le N-1$

$$\frac{dx_{ii}}{dt} = -\frac{dy_i}{2\,dt} = \frac{1}{2}e^{(x_i - x_{i+1})} - \frac{1}{2}e^{(x_{i-1}-x_i)} \tag{60.1}$$
$$= 2x_{ii+1}^2 - 2x_{i-1,i}^2$$
$$= 2\left(x_{ii+1}^2 - x_{i-1\,i}^2\right)$$

For $i=1$, $i=N$

$$\frac{dx_{11}}{dt} = 2x_{12}^2, \quad \frac{dx_{NN}}{dt} = -2\cdot x_{N-1\,N}^2 \tag{60.2}$$

Or more uniformly,

(61.0) $$\frac{dx_{ii}}{dt} = 2\left(x_{i\,i+1}^2 - x_{i-1,i}^2\right), \quad 1 \le i \le N, \quad \text{where } x_{N\,N+1} = x_{01} \equiv 0.$$

For $1 \le i \le N-1$

(61.1) $$\frac{dx_{i+1\,i}}{dt} = \frac{dx_{i,i+1}}{dt} = \frac{1}{4} e^{\frac{1}{2}(x_i - x_{i+1})}\left(\frac{dx_i}{dt} - \frac{dx_{i+1}}{dt}\right)$$

$$= \frac{1}{2} x_{i\,i+1}\left(y_i - y_{i+1}\right)$$

$$= - x_{i\,i+1}\left(x_{ii} - x_{i+1,i+1}\right)$$

$$= x_{i\,i+1}\left(x_{i+1,i+1} - x_{ii}\right)$$

Let $B(X) = X_- - X_-^T$ where X_- is the strictly lower triangular part of X. Then, using $x_{i\,i+1} = x_{i+1\,i}$

(61.2) $$B(X) = -B(X)^T = \begin{pmatrix} 0 & -x_{21} & & & \\ x_{21} & 0 & -x_{32} & & \\ & x_{32} & 0 & \ddots & \\ & & \ddots & \ddots & -x_{N,N-1} \\ & & & x_{N,N-1} & 0 \end{pmatrix}$$

Now compute the commutator

(61.3) $$[X, B(X)] = XB - BX = XB + (XB)^T$$

We find

As X is tridiagonal, $X_{ik} = 0$ for $|i-k| > 1$,

$$(XB)_{ij} = \sum_{k=1}^{N} X_{ik} B_{kj} = X_{i\,i-1} B_{i-1,j} + X_{ii} B_{ij} + X_{i\,i+1} B_{i+1\,j}$$

Now for $|i-j| > 2$, $|i\pm 1 - j| \geq |i-j| - 1 > 1$, and hence, as B is tridiagonal, $B_{i-1\,j} = B_{ij} = B_{i+1\,j} = 0$. Thus $(XB)_{ij} = 0$

(62.1) Thus $(XB)_{ij} = 0$

If $j = i-2$, $B_{i-1\,j} = B_{i-1\,i-2} = X_{i-1,i-2}$

$B_{i\,j} = B_{i\,i-2} = 0$

$B_{i+1\,j} = B_{i+1\,i-2} = 0$

(62.2) Thus $(XB)_{i\,i-2} = X_{i\,i-1} X_{i-1\,i-2}$

If $j = i-1$, $B_{i-1\,j} = B_{i-1\,i-1} = 0$

$B_{i\,j} = B_{i,i-1} = X_{i\,i-1}$

$B_{i+1\,j} = B_{i+1\,i-1} = 0$

(62.3) Thus $(XB)_{i\,i-1} = X_{ii} X_{i\,i-1}$

If $j = i$, $B_{i-1\,j} = B_{i-1\,i} = -X_{i\,i-1}$

$B_{i\,j} = B_{ii} = 0$

$B_{i+1\,j} = B_{i+1\,i} = X_{i+1\,i}$

(62.4) Thus $(XB)_{ii} = -X_{i\,i-1}^2 + X_{i\,i+1} X_{i+1\,i} = -X_{i\,i-1}^2 + X_{i+1\,i}^2$

If $j = i+1$, $B_{i-1\,j} = B_{i-1\,i+1} = 0$

$B_{i\,j} = B_{i\,i+1} = -X_{i+1\,i}$

$B_{i+1\,j} = B_{i+1\,i+1} = 0$

(62.5) Thus $(XB)_{i\,i+1} = -X_{ii} X_{i+1\,i}$

If $j = i+2$, $B_{i-1\,j} = B_{i-1\,i+2} = 0$

$B_{i\,j} = B_{i\,i+2} = 0$

$B_{i+1\,j} = B_{i+1\,i+2} = -X_{i+2\,i+1}$

(62+.1) Thus $(XB)_{i\,i+2} = -X_{i\,i+1} X_{i+2\,i+1} = -X_{i+1\,i} X_{i+2\,i+1}$

Now from (61.3)

$$[X,B]_{ij} = (XB)_{ij} + (XB)_{ji}$$

If $|i-j| > 2$ we have

(62+.2) $$[X,B]_{ij} = 0$$

If $j = i-2$, we have from (62.2) and (62+.1)

$[X,B]_{ij} = [X,B]_{i\,i-2} = (XB)_{i\,i-2} + (XB)_{i-2\,i}$

(62+.3) $= X_{i\,i-1}X_{i-1\,i-2} - X_{i-1\,i-2}X_{i\,i-1} = 0$

If $j = i-1$, we have from (62.3) and (62.5)

$[X,B]_{ij} = [X,B]_{i\,i-1} = (XB)_{i\,i-1} + (XB)_{i-1\,i}$

(62+.4) $= X_{ii}X_{i\,i-1} - X_{i-1\,i-1}X_{i\,i-1} = X_{i\,i-1}(X_{ii} - X_{i-1\,i-1})$

If $j = i$, we have from (62.4)

$[X,B]_{ij} = [X,B]_{ii} = (XB)_{ii} + (XB)_{ii}$

(62+.5) $= 2(X^2_{i+1\,i} - X^2_{i\,i-1})$

If $j = i+1$, we have as $[X,B]$ is symmetric

(62+.6) $[X,B]_{ij} = [X,B]_{i\,i+1} = [X,B]_{i+1\,i} = X_{i+1\,i}(X_{i+1\,i+1} - X_{ii})$

and if $j = i+2$

$[X,B]_{ij} = [X,B]_{i\,i+2} = [X,B]_{i+2} = 0$

In particular $[X,B]$ is tridiagonal and comparing (61.0) with (62+.5) and (61.1) with (62+.4) we obtain the Lax-Pair formulation of Toda's lattice equation

$(x(t), y(t))$ solves (59.2), Toda's equations

$$\Longleftrightarrow$$

$$\frac{dX}{dt} = [X, B(X)] \quad , \quad X(t=0)\ X_0$$

where X is given by (59.2) and $B(X) = X_- - X_-^T$, and

Insert 63+

X_0 is expressed in terms of $(x(t=0), y(t=0))$.

It is conventional to write

(63.1)
$$\begin{cases} a_i = X_{ii} = -y_i/2 & , 1 \le i \le N \\ b_i = X_{i\,i+1} = X_{i+1,i} = \frac{1}{2} e^{(x_i - x_{i+1})} & , 1 \le i \le N-1 \end{cases}$$

So that

$$X = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & \ddots & \ddots & 0 \\ & \ddots & \ddots & b_{N-1} \\ 0 & & b_{N-1} & a_N \end{pmatrix} , \quad B(X) = \begin{pmatrix} 0 & -b_1 & & \\ b_1 & \ddots & & 0 \\ & & \ddots & \\ 0 & & \ddots & -b_{N-1} \\ & & b_{N-1} & 0 \end{pmatrix}$$

And the Toda equations have the form

(63.2)
$$\frac{da_i}{dt} = 2\left(b_i^2 - b_{i-1}^2\right) \quad , \quad i = 1, \ldots, N$$
$$\frac{db_i}{dt} = b_i\left(a_{i+1} - a_i\right) \quad , \quad i = 1, \ldots, N-1$$

(63+.1) Remark

Note that the elements in the diagonals $|i-j|=2$ of $[X,B]$ are automatically zero, so that $[X,B]$ is tri-diagonal if X is tridiagonal. In the above analysis, this appears to be just a matter of calculation. But it is easy to see that this has a structural reason. Indeed,

$$B(X) = \begin{pmatrix} 0 & -x_{21} & & & 0 \\ x_{21} & 0 & -x_{32} & & \\ & x_{32} & \ddots & \ddots & \\ & & \ddots & & -x_{N\,N-1} \\ 0 & & & x_{N\,N-1} & 0 \end{pmatrix}$$

$= X + H$ where H is upper triangular

Hence

$$[X, B(X)] = [X, X] + [X, H] = XH - HX$$

It is now easy to see that XH and HX have only 1 non-zero diagonal below the main diagonal. But $[X, B(X)]$ is symmetric. Hence $[X, B(X)]$ is tri-diagonal

(63+.2) Exercise Use the above argument show that if X is finite banded, and $B(X) = X_- - X_-^T$, then $[X, B(X)]$ has the same band structure as X. Thus $\dot{X} = [X, B(X)]$ is a band-preserving flow (more later).

where $b_0 \equiv 0$, $b_N \equiv 0$.

$$H_T(x,a) = 2 \sum_{i=1}^{N} a_i^2 + 4 \sum_{i=1}^{N-1} b_i^2 \;)$$

$$= 2 \left(\sum_{i=1}^{N} a_i^2 + 2 \sum_{i=1}^{N} b_i^2 \right.$$

i.e

(64.1) $$H_T(x,a) = 2 \operatorname{tr} X^2$$

As $\operatorname{tr} X^2 = \frac{1}{2} H_T(x,a)$ is conserved, we have, as noted in lecture 1, ~~we have~~ an a priori bound on solutions of (63.2), $\operatorname{tr} X(t)^2 = \operatorname{tr} X(0)^2$, so the Toda equations have global existence. Note that as

$$b_i^2(t) = \frac{1}{4} e^{(x_i - x_{i+1})} \le \operatorname{tr} X(0)^2$$

we see, a priori, that

(64.2) $$x_i(t) \le x_{i+1}(t) + c_i, \quad \text{for some } c_i < \infty$$

so that particle i cannot get too far ahead of particle $i+1$. We will see in fact that the particles will order themselves as $x_1(t) < x_2(t) < \dots < x_N(t)$, as $t \to +\infty$.

Note also that a direct proof of global existence for the Toda equations in (x, y) variables, proceeds by noting that

$$\frac{1}{2} \sum_{i=1}^{N} y_i^2 \leq H_T(x, y) = \text{const}.$$

so $\quad |y_i(t)| \leq 2 H_T$

$$\Rightarrow \quad x_i(t) = x_i(0) + \int_0^t y_i(s)\, ds$$

$$\Rightarrow \quad |x_i(t)| \leq |x_i(0)| + 2 H_T\, t$$

from which standard ODE arguments imply global existence, even though the x_i's, can, and in fact do, grow (linearly) as $t \to \infty$.

As noted in lecture 1, the Lax pair form of the Toda equations imply that

$$\text{spec}\, X(t) = \text{const} = \text{spec}\, X(0)$$

Thus

(66.1) $$\lambda_k(t) = \lambda_k(0)$$

where the λ_k's are the eigenvalues of $X(t)$.

We will show later on that they are independent and Poisson commute. Our immediate goal is just

~~The following result is of basic interest.~~

~~**Lemma:** Let~~ $M = \{(\lambda_1, \lambda_2, \dots, \lambda_N ; x_1, \dots, x_N) : \lambda_1 > \lambda_2 > \dots > \lambda_N, \; x_i > 0, \; \sum_{i=1}^{N} x_i^2 = 1\}$

~~Then the map~~

$(a_1, \dots, a_N, b_1, \dots, b_{N-1}) \longmapsto \lambda_1, \dots, \lambda_N, u$

to solve the Toda equations explicitly. We need a number of preliminary results which are of general interest.

Definition A real, symmetric tri-diagonal matrix with strictly positive off-diagonal entries is called a Jacobi matrix

Lemma 66.1 Let $X = \begin{pmatrix} a_1 & b_1 & & 0 \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{N-1} \\ 0 & & b_{N-1} & a_N \end{pmatrix}$ be a Jacobi matrix.

Then the spectrum of X is simple and the first (and last) component of eigenvectors of X are non-zero i.e. if $Xu = \lambda u$, $u = (u_1, \dots, u_N)^T \neq 0$, then $u_1 \neq 0$.

Proof: Let $u = (u_1, \dots, u_N)^T \neq 0$ be an eigenvector associated with an eigenvalue λ of X. Then

$$\begin{pmatrix} a_1-\lambda & b_1 & & \\ b_1 & a_2-\lambda & b_2 & \\ & \ddots & \ddots & b_{N-1} \\ & & b_{N-1} & a_N-\lambda \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} = 0$$

which implies

$$(a_1-\lambda)u_1 + b_1 u_2 = 0$$
$$b_1 u_1 + (a_2-\lambda) u_2 + b_2 u_3 = 0$$
$$\vdots$$
$$b_{k-1} u_{k-1} + (a_k-\lambda) u_k + b_k u_{k+1} = 0$$
$$\vdots$$
$$b_{N-1} u_{N-1} + (a_N-\lambda) u_N = 0$$

Thus if $u_1 = 0$, we see from the first equation that, as $b_1 \neq 0$, $u_2 = 0$. But then from the second equation $u_3 = 0$, etc and so $u = 0$, which is a contradiction. So $u_1 \neq 0$

Now if λ is not a simple eigenvalue, then there exist at least two eigenvectors, $(X - \lambda) u = 0$, $(X - \lambda)\hat{u} = 0$

As $u_1 \neq 0$, we see that for suitable α, $w = \alpha u - \hat{u}$ is an eigenvector of X corresponding to λ with $w_1 = 0$.

But then $w = 0$ and so $\hat{u} = \alpha u$. Thus the spectrum of X is simple. $\square$

Let $\tilde{J} = \tilde{J}_N$ denote the collection of $N \times N$ Jacobi matrices. For a matrix $X \in \tilde{J}$, we may order the eigenvalues

(68.1) $$\lambda_1 > \lambda_2 > \dots > \lambda_N$$

and (uniquely) specify the associated normalized eigenvectors

(68.2) $u = (u_1, \dots, u_N)^T$, $\sum_{i=1}^{N} u_i^2 = 1$, by requiring $u_1 > 0$.

With these specifications we have the following basic result.

Let $M = \{(\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N) : \alpha_1 > \dots > \alpha_N, \sum_{i=1}^{N} \beta_i^2 = 1, \beta_i > 0\} \subset \mathbb{R}^{2N}$. (69)

Lemma 69.1 Let $N \geq 1$.

The map $\varphi : J \to M$ taking a Jacobi X

to its eigenvalues $\lambda_1 > \lambda_2 \dots > \lambda_N$ and the first components

of its normalized eigenvectors $(u_{1 1}, \dots, u_{1N})$,

$$(X - \lambda_j) u_j = 0 \quad , \quad u_j = (u_{1j}, \dots, u_{N,j})^T, \ \|u_j\| = 1, \ 1 \leq j \leq N$$

is a (well-defined) diffeomorphism from $\tilde{J}$ onto M.

Proof: By the spectral theorem

(69.2) $$X = U \Lambda U^T$$

where $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_N)$, $\lambda_1 > \dots > \lambda_N$, and $U = (u_{ij})$

is the orthogonal matrix of the associated normalized eigenvectors.
with $(u_{1j}, \dots, u_{Nj})^T$ the eigenvector corresponding to λ_j.
From (69.2) we have, using the standard basis $\{e_i\}_{i=1}^{N}$, for any $i \in \{1, \dots, N\}$,

(69.3) $$a_i = X_{ii} = (e_i, U \Lambda U^T e_i) = (U^T e_i, \Lambda U^T e_i)$$

$$= \left(\begin{pmatrix} u_{i1} \\ \vdots \\ u_{iN} \end{pmatrix}, \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} \begin{pmatrix} u_{i1} \\ \vdots \\ u_{iN} \end{pmatrix} \right) = \sum_{j=1}^{N} \lambda_j u_{ij}^2$$

and for $1 \le i \le N-1$

(70.0) $$b_i = X_{i\,i+1} = (U^T e_i, \Lambda U^T e_{i+1})$$

$$= \left(\begin{pmatrix} u_{i1} \\ \\ u_{iN} \end{pmatrix}, \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} \begin{pmatrix} u_{i+1\,1} \\ \vdots \\ u_{i+1\,N} \end{pmatrix} \right)$$

$$= \sum_{j=1}^{N} \lambda_j u_{ij} u_{i+1\,j}$$

From the eigenvalue equation

$$b_i u_{i+1,j} = (\lambda_j - a_i) u_{ij} - b_{i-1} u_{i-1,j}, \quad 1 \le i \le N, \quad 1 \le j \le N$$

$(b_0 = b_N \equiv 0)$ and the orthogonality of U, it follows that

(70.1) $$b_i^2 = \sum_{j=1}^{N} \left((\lambda_j - a_i) u_{ij} - b_{i-1} u_{i-1,j} \right)^2, \quad 1 \le i \le N-1$$

and so

(70.2) $$u_{i+1\,j} = \frac{1}{b_i} \left((\lambda_j - a_i) u_{ij} - b_{i-1} u_{i-1,j} \right), \quad i = 1, \dots, N-1.$$

These facts immediately imply that φ is 1-1.

Indeed, (from (69.3)) $a_1 = \sum_{j=1}^{N} \lambda_j u_{1j}^2$ is clearly determined by the λ_i's and the u_{1j}'s. Then from (70.1)

$$b_1^2 = \sum_{j=1}^{N} \left((\lambda_j - a_1) u_{1j} - 0 \right)^2$$

and

$$u_{2j} = ((\lambda_j - a_1)u_{1j} - 0)/b_1$$

we see that $b_1 (>0)$ and u_{2j} are determined.

Continuing, we have

$$a_2 = \sum_{j=1}^{N} \lambda_j u_{2j}^2$$

and so a_2 is now determined, and

$$b_2^2 = \sum_{j=1}^{N} ((\lambda_j - a_2) u_{2j} - b_1 u_{1j})^2$$

$$u_{3j} = ((\lambda_j - a_2) u_{2j} - b_1 u_{1j})/b_2$$

and so $b_2 (>0)$ and u_{3j} are determined. We conclude

by a simple induction that φ is 1-1.

Conversely given a point $(\alpha_1, .., \alpha_N, \beta_1, .., \beta_N)$ in X_q
Set $u_{1j} = \beta_j$, $j = 1, .., N$ and

(71.1) $$a_1 = \sum_{j=1}^{N} \alpha_j u_{1j}^2$$

(71.2) $$b_1^2 = \sum_{j=1}^{N} ((\alpha_j - a_1) u_{1j})^2.$$

As the α_j's are distinct $(N>1)$, we see that $b_1^2 \neq 0$, and (S.O.) we may choose $b_1 > 0$
Set

(71.3) $$u_{2j} = (\alpha_j - a_1) u_{1j} / b_1$$

Then clearly

$$(a_1 - \alpha_j) u_{1j} + b_1 u_{2j} = 0$$

Now $\sum_{j=1}^{N} (a_1 - \alpha_j) u_{1j}^2 + b_1 \sum_{j=1}^{N} u_{2j} u_{1j} = a_1 - \sum_{j=1}^{N} \alpha_j u_{1j}^2 + b_1 \sum_{j=1}^{N} u_{2j} u_{2j}$

and hence by (71.1) and $b_1 > 0$, we conclude that $u^{(1)} = (\beta_1, \dots, \beta_N)^T = (u_{11} \dots u_{1N})^T$ and $u^{(2)} = (u_{21}, \dots, u_{2N})^T$ are orthogonal unit vectors.

Set
$$a_2 \equiv \sum_{j=1}^{N} \alpha_j u_{2j}^2$$

$$b_2^2 \equiv \sum_{j=1}^{N} \left[(\alpha_j - a_2) u_{2j} - b_1 u_{1j} \right]^2, \quad b_2 > 0$$

and

$$u_{3j} = \left((\alpha_j - a_2) u_{2j} - b_1 u_{1j} \right) / b_2$$

Notice that if $b_2 = 0$, then

$$b_1 \beta_j + (a_2 - \alpha_j) u_{2j} = 0, \quad j = 1, \dots, N$$

In other words

$$\begin{pmatrix} a_1 - \alpha_j & b_1 \\ b_1 & a_2 - \alpha_j \end{pmatrix} \begin{pmatrix} \beta_j \\ u_{2j} \end{pmatrix} = 0, \quad j = 1, \dots, N$$

and as $\beta_j \neq 0$, we see that the matrix

$$\begin{pmatrix} a_1 & b_1 \\ b_1 & a_2 \end{pmatrix}$$

has N distinct eigenvalues $\alpha_1, \dots, \alpha_N$. If $N > 2$, this is a contradiction.

Now observe from (71.2) (71.3)

$$b_1^2 = \sum_{j=1}^{N} (\alpha_j - a_1) u_{1j} \; b_1 u_{2j} = b_1 \sum_{j=1}^{N} \alpha_j u_{1j} u_{2j}$$

i.e.

(73.1) $$b_1 = \sum_{j=1}^{N} \alpha_j u_{1j} u_{2j} \qquad (\text{cf } (70.0)).$$

It follows that

$$\sum_{j=1}^{N} u_{3j} u_{1j} = \sum_{j=1}^{N} \left((\alpha_j - a_2) u_{2j} u_{1j} - b_1 u_{1j}^2 \right) / b_2$$

$$= \left(\sum_{j=1}^{N} \alpha_j u_{1j} u_{2j} - b_1 \right) / b_2$$

$$= 0 \quad \text{by } (73.1)$$

Also

$$\sum_{j=1}^{N} u_{3j} u_{2j} = \sum_{j=1}^{N} \left[(\alpha_j - a_2) u_{2j}^2 - b_1 u_{1j} u_{2j} \right]$$

$$= \sum_{j=1}^{N} \alpha_j u_{2j}^2 - a_2 - 0$$

$= 0$ by the definition of a_2

By induction assume that for $1 \leq i \leq k$ we have

k unit orthogonal vectors $u^{(i)} = (u_{i1}, \dots, u_{iN})^T$, $1 \leq i \leq k$;

together with $k-1$ real scalars $a_1, \dots, a_{k-1}$, $b_1, \dots, b_{k-1}$ with

$b_i > 0$, $i = 1, \ldots, k$ such that

(74.1) $$b_i u_{i+1\, j} + (a_i - \alpha_j) u_{ij} + b_{i-1} u_{i-1\, j} = 0 \quad , \quad 1 \le j \le N \, , \, b_0 = 0$$

for $1 \le i \le k-1$

As $(u^{(i)}, u^{(i')}) = \delta_{ii'}$, $1 \le i, i' \le k$, we must have

$$b_i (u^{(i+1)}, u^i) + a_i (u^{(i)}, u^{(i)}) - \sum_{j=1}^{N} \alpha_j u_{ij}^2 + b_{i-1} (u^{(i-1)}, u^i) = 0$$

and so

(74.2) $$a_i = \sum_{j=1}^{N} \alpha_j u_{ij}^2 \quad , \quad 1 \le i \le k-1.$$

Also

$$b_i (u^{(i+1)}, u^{(i+1)}) + a_i (u^{(i)}, u^{(i+1)}) - \sum_j \alpha_j u_{ij} u_{i+1\, j} + b_{i-1} (u^{(i-1)}, u^{(i+1)}) = 0$$

and so

(74.3) $$b_i = \sum_{j=1}^{N} \alpha_j u_{ij} u_{i+1\, j} \quad , \quad 1 \le i \le k-1$$

Set

(74.4) $$a_k = \sum_{j=1}^{N} \alpha_j u_{kj}^2$$

and set

(74.5) $$b_k^2 = \sum_{j=1}^{N} \left[(a_k - \alpha_j) u_{kj} + b_{k-1} u_{k-1\, j} \right]^2$$

If $k \le N-1$, we must have $b_k^2 \neq 0$. Indeed, if

$b_k^2 = 0$, then

$$(a_k - \alpha_j) u_{kj} + b_{k-1} u_{k-1\,j} = 0 \qquad 1 \le j \le N$$

Together with (74.1), we see that $(u_{1j}, \dots, u_{kj})^T$ is an eigenvector of the Jacobi matrix

$$\begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & & \\ & & \ddots & & b_{k-1} \\ & & & b_{k-1} & a_k \end{pmatrix}$$

with eigenvalue α_j for $1 \le j \le N$. As the α_j's are distinct, and as $k \le N-1$, this is a contradiction. Hence $b_k^2 > 0$ and we choose $b_k > 0$.

Let

$$u_{k+1,j} \equiv \left((\alpha_j - a_k) u_{kj} + b_{k-1} u_{k-1\,j} \right) / b_k$$

and $u^{(k+1)} = (u_{k+1,1}, \dots, u_{k+1,N})^T$. Clearly $\| u^{(k+1)} \| = 1$.

We have

(75.1) $$b_k u_{k+1,j} + (a_k - \alpha_j) u_{kj} + b_{k-1} u_{k-1\,j} = 0$$

Now

(75.2) $$b_k (u^{(k+1)}, u^{(k-1)}) + a_k (u^{(k)}, u^{(k-1)}) - \sum_{j=1}^{N} \alpha_j u_{kj} u_{k-1\,j} + b_{k-1} = 0$$

Now from (74.3), $b_{k-1} = \sum_{j=1}^{N} \alpha_j u_{kj} u_{k-1\,j}$. As $b_k > 0$,

it follows from (75.2) that $(u^{(k+1)}, u^{(k-1)}) = 0$

We also have from (75.1)

$$b_k (u^{(k+1)}, u^{(k)}) + a_k (u^{(k)}, u^{(k)}) - \sum \alpha_j u_{kj}^2 + b_{k-1} (u^{(k-1)}, u^{(k)}) = 0$$

and it follows from (74.4) that $(u^{(k+1)}, u^{(k)}) = 0$

Clearly $(u^{(k+1)}, u^{(i)}) = 0$ for $i < k-1$.

Thus we have shown that if $k \le N-1$ we

have $k+1$ orthonormal eigenvectors $u^{(i)}$, $1 \le i \le k+1$ together

with k real numbers $a_1, \dots, a_k, b_1, \dots, b_k$ with $b_i > 0$, $i = 1, \dots, k$

such that

(76.1) $$b_i u_{i+1\,j} + (a_i - \alpha_j) u_{ij} + b_{i-1} u_{i-1\,j} = 0 \quad , \quad 1 \le j \le N$$

and $1 \le i \le k$. This verifies the induction for all

$1 \le k \le N-1$.

For $k = N$, set

(76.2) $$a_N = \sum_{j=1}^{N} \alpha_i u_{Nj}^2$$

Now necessarily,

$$(a_N - \alpha_j)\, u_{Nj} + b_{N-1}\, u_{N-1\,j} = 0 \quad , 1 \le j \le N$$

If not

$$v_j \equiv (a_N - \alpha_j)\, u_{Nj} + b_{N-1}\, u_{N-1,j} \quad , \quad 1 \le j \le N$$
$$v = (v_1, \dots, v_N)^T$$

would be a non-zero vector which is orthogonal to $u^{(i)}$,

$i = 1, \dots, N$. Indeed

$$(v, u^{(N)}) = a_N - \sum \alpha_j u_{Nj}^2 + b_{N-1}(u^{N-1}, u^N)$$
$$= 0 \quad , \text{by } (76.2)$$

and

$$(v, u^{(N-1)}) = a_N (u^{(N)}, u^{(N-1)}) - \sum \alpha_j u_{Nj} u_{N-1}$$
$$+ b_{N-1} (u^{N-1}, u^{N-1})$$

which is (74.3). Clearly $(v, u^{(i)}) = 0$, $1 \le i < N-1$.

It follows that for $1 \le j \le N$

$$\begin{pmatrix} a_1 - \alpha_j & b_1 & & \\ b_1 & a_2 - \alpha_j & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & b_{N-1} \\ & & b_{N-1} & a_N - \alpha_j \end{pmatrix} \begin{pmatrix} u_{1j} \\ \vdots \\ \vdots \\ u_{Nj} \end{pmatrix} = 0$$

Thus $X \equiv \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & & \\ & & \ddots & b_{N-1} \\ & & b_{N-1} & a_N \end{pmatrix}$ is a Jacobi matrix with eigenvalues $\lambda_1 = \alpha_1 > \lambda_2 = \alpha_2 > \dots > \lambda_N = \alpha_N$

and with eigenvectors $(u_{1j}, \dots, u_{Nj})^T$ corresponding to λ_j,

whose first components are $u_{1j} = \beta_j > 0$, $j = 1, \dots, N$.

This shows that $\varphi : \tilde{J} \to M$ is bijective. Clearly

φ and φ^{-1} are smooth so that φ is a diffeomorphism. $\square$