Lecture 10 In these lectures we are interested in using the Toda algorithm to compute the top eigenvalue of a random real symmetric or Hermitian matrix. We do this by running the algorithm until $E(t) < \varepsilon$. As noted above this tells us that $|X_{11}(t) - \lambda_j| < \varepsilon$ for some eigenvalue λ_j of $X(t=0)$. So in order to conclude that λ_j is typically the top eigenvalue, we need to know that $X_{11}(t)$ necessarily converges to the top eigenvalue with high probability.

From the calculations on pp 143 et seq, we see that $u_{1j}(0) \neq 0$ for some $1 \le j \le m$ where

$$\lambda_1 = \lambda_2 = \cdots = \lambda_m > \lambda_{m+1} \geq \lambda_{m+2} \geq \cdots \geq \lambda_N$$

is a sufficient condition for $X_{11}(t) \to \lambda_1$.

Let $M \in \Gamma_N$ be an $N \times N$ Hermitian matrix. Suppose

M has eigenvalue $\{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N\}$ and associated orthonormal eigenvectors $u(\lambda_j) = (u_{1j}, \dots, u_{Nj})^T$. Let

$$A_N = \{M \in \Gamma_N : \text{for all eigenvectors } u = (u_1, \dots, u_N)^T \text{ of } M,\ u_1 \neq 0\}.$$

Clearly if $M \in A_N$ and $X(t)$ solves the extended Toda flow (135.1) with $X(0) = M$, then $X_{11}(t) \to \lambda_1$ as $t \to \infty$. Thus $M \in A_N$ is a sufficient condition for the Toda algorithm to compute the top eigenvalue of M.

The following result shows that $M \in A_N$ is a generic condition in a strong sense.

<u>Theorem 149.1</u>

A_N is a dense open set in Γ_N of full measure ie meas$(\Gamma_N \setminus A_N) = 0$

Note first that $A_N \subset \{M \in \Gamma_N : \lambda_j(M) \neq \lambda_k(M), j \neq k\}$

Indeed if $M \in A_N$ and $\lambda_j(M) = \lambda_k(M) \equiv \lambda$, then associated with λ there are at least two independent eigenvectors $M u(\lambda_j) = \lambda u(\lambda_j)$, $M u(\lambda_k) = \lambda u(\lambda_k)$, from which it follows that there is an eigenvector $u \neq 0$, $Mu = \lambda u$ with $u(1) = 0$, which is a contradiction. Hence $M \in A_N$ has simple eigenvalues.

Let $C_N = A_N' = \Gamma_N \setminus A_N$.

For an $N \times N$ matrix M, let M_1 denote the $N-1 \times N-1$ matrix obtained by deleting the first row and column of M.

<u>Claim</u>

$$C_N = D_N = \{M \in \Gamma_N : M \text{ and } M_1 \text{ have a common eigenvalue}\}$$

<u>Proof</u>: Clearly $C_N \subset D_N$ for if $Mu = \lambda u$, $u = (u_1, \dots, u_N)$ with $u_1 = 0$, then $v = (u_2, \dots, u_N)^T \neq 0$ is clearly an eigenvector of M_1, $M_1 v = \lambda v$. Conversely, suppose that $M \in D_N$, but $M \notin C_N$. Let λ be a common eigenvalue

of M and M_1. Then $(M-\lambda)u = 0$ for some

$u = (u_1, \dots, u_N)^T \neq 0$ and as $M \notin C_N$, $u_1 \neq 0$. Without

loss, suppose $u_1 = -1$. Write

$$M - \lambda = \begin{pmatrix} a & b^* \\ b & M_1 - \lambda \end{pmatrix}$$

where $a \in \mathbb{R}$ and $b \in \mathbb{C}^{N-1}$. As $(M-\lambda)u = 0$, we

have in particular

$$b = (M_1 - \lambda)(u_2, \dots, u_N)^T.$$

and so

$$\begin{aligned} b^* &= (\bar{u}_2, \dots, \bar{u}_N)(M_1 - \lambda)^* \\ &= (\bar{u}_2, \dots, \bar{u}_N)(M_1 - \lambda) \end{aligned}$$

Now there exists $w = (w_2, \dots, w_N)^T \neq 0$ such that

$(M_1 - \lambda)w = 0$, and so $b^* w = 0$. But then

$$(M - \lambda)\begin{pmatrix} 0 \\ w \end{pmatrix} = \begin{pmatrix} a & b^* \\ b & M_1 - \lambda \end{pmatrix}\begin{pmatrix} 0 \\ w \end{pmatrix} = 0$$

and so M has an eigenvector $u = \begin{pmatrix} 0 \\ w \end{pmatrix}$ with $u_1 \neq 0$. This

is a contradiction and so $D_N \subset C_N$. This proves the Claim. $\square$

<u>Remark 1</u>. By the proof of the Claim, we in fact see that if $M \in D_N$, then M and M_1 have a "common" eigenvector i.e. an eigenvector $w = (w_2, \ldots, w_N)^T$ for M_1, $M_1 w = \lambda w$, such that $v = (0, w_2, \ldots, w_N)^T$ is an eigenvector for M, $Mv = \lambda v$. Note however, that not <u>every</u> eigenvector w of M_1 has the property that $v = \binom{0}{w}$ is an eigenvector for M. For example, if

$$M = \begin{pmatrix} a & b & c \\ \bar{b} & 0 & 0 \\ \bar{c} & 0 & 0 \end{pmatrix}, \quad |b|^2 + |c|^2 \neq 0$$

Then $M_1 = 0$ and $(u_2, u_3)^T$ is an eigenvector for M_1, for every u_2, u_3, $|u_2|^2 + |u_3|^2 \neq 0$. However $(0\ u_2\ u_3)^T$ is an eigenvector of M if and only if $(u_2, u_3)^T$ is a multiple of $(-c\ b)^T$.

We now show that D_N has measure 0 in $\Gamma_N \cong \mathbb{R}^{N^2}$.

Let

$$p(\lambda) = \det(\lambda - M), \qquad q(\lambda) = \det(\lambda - M_1).$$

Write

$$p(\lambda) = \lambda^N + p_{N-1}\lambda^{N-1} + \dots + p_0$$
$$q(\lambda) = \lambda^{N-1} + q_{N-2}\lambda^{N-2} + \dots + q_0.$$

and consider the <u>resultant</u> R of p and q.

(153.1)
$$R = \det \begin{pmatrix} p_0 & p_1 & \dots & p_{N-1} & 1 & & & 0 \\ & p_0 & & & p_{N-1} & 1 & & \\ 0 & & \ddots & & & & \ddots & \\ & & & p_0 & \dots & \dots & \dots & 1 \\ q_0 & q_1 & \dots & q_{N-2} & 1 & & & \\ & q_0 & \dots & \dots & & 1 & & 0 \\ 0 & & \ddots & & & & \ddots & \\ & & & q_0 & \dots & \dots & \dots & 1 \end{pmatrix}$$

where $(p_0 \dots p_{N-1}\ 1)$ is repeated $N-1$ times

and $(q_0, \dots, q_{N-2}, 1)$ is reapeated N times

Clearly R is a determinant of size $N + (N-1) = 2N-1$.

(153.2) <u>(Standard) Exercise</u> $R = 0 \iff p(\lambda)$ and $q(\lambda)$ have a common root

See also Example 154.1 below.

Now $R = R(M)$ is a real analytic function (in fact a polynomial) in the N^2 real entries of M and hence if it vanishes on a set of positive measure in $\mathbb{R}^{N^2} \cong \Gamma_N$, it is identically zero. This is also a standard exercise. But

$$M = \begin{pmatrix} 1 & 1 & & \cdots & 1 \\ 1 & 2 & & & \\ & & 3 & 0 & \\ \vdots & 0 & & & \\ 1 & & & & N \end{pmatrix} = M^* = M^T$$

does not share a common eigenvalue with $M_1 = \mathrm{diag}(2, 3, \ldots, N)$, spec $M_1 = \{2, 3, \ldots, N\}$. Indeed if $Mu = ju$, $u = (u_1, \ldots, u_N) \neq 0$, for some $2 \leq j \leq N$, then $0 = (e_j, (M-j)u) = u_1$. But then $(k-j)u_k = 0$, for $2 \leq k \leq n$, $k \neq j$, and so

$$u = (0, \ldots, 0, u_j, 0, \ldots, 0)^T$$

But from the first row of $(M-j)$, we see that $u_j = 0$, so $u = 0$. We conclude $R \not\equiv 0$. Hence we must have meas D_n = meas $A_n' = 0$

This proves Theorem 149.1 $\square$.

Example 154+1.1

Suppose $p(\lambda) = p_0 + p_1\lambda + \lambda^2$, $q(\lambda) = q_0 + \lambda$

Then R is a determinant of size $2N-1 = 4-1 = 3$,

$$R = \det \begin{pmatrix} p_0 & p_1 & 1 \\ q_0 & 1 & 0 \\ 0 & q_0 & 1 \end{pmatrix} = p_0 - p_1 q_0 + q_0^2$$

So if $\lambda = -q_0$ is a root of q, and also a root

of $p(\lambda) = p_0 - q_0 p_1 + q_0^2$, we see that $R = 0$.

Now as noted earlier, the Toda algorithm

is not necessarily ordering on general full matrices

in Σ_N or T_N. We have shown that if $X_0 \in T_N$,

and $X(t)$ solves the Toda flow (135.1) with $X(0) = X_0$, then

$X_{11}(t) \to \lambda_1$. We now describe a condition on X_0

that guarantees that $X_{22}(t) \to \lambda_2$.

Let $u_i(t)$ be a smooth eigenvector for $X(t)$,

$X(t) u_i(t) = \lambda_i u_i(t)$, constructed as above to satisfy.

$$\frac{d}{dt} u_j(t) + \tilde{B}(X(t))\, u_j(t) = 0 \tag{155.1}$$

as in (141.1) above. We conclude as above that

$$\dot{u}_{1j}(t) = (\lambda_j - (X(t))_{11})\, u_{1j}(t)$$

Now

$$\dot{u}_{2j}(t) = -(e_2, \tilde{B}(X(t))\, u_j(t))$$

$$= (\tilde{B}(X(t))\, e_2, u_j(t))$$

$$= (X(t) e_2 - X_{22}(t) e_2 + X_{21} e_1, u_j(t))$$

$$= ((\lambda_j - X_{22}(t)) e_2 + X_{21} e_1, u_j(t))$$

$$= (\lambda_j - X_{22}(t))\, u_{2j}(t) + X_{21}\, u_{1j}(t)$$

Thus for $j \neq k$

$$\frac{d}{dt}\left(u_{1j}(t)\, u_{2k}(t) - u_{1k}(t)\, u_{2j}(t)\right)$$

$$= (\lambda_j - (X)_{11})\, u_{1j} u_{2k} + u_{1j}\left((\lambda_k - X_{22}) u_{2k} + X_{21} u_{1k}\right)$$

$$- (\lambda_k - (X)_{11})\, u_{1k} u_{2j} - u_{1k}\left((\lambda_j - X_{22}) u_{2j} + X_{21} u_{1j}\right)$$

$$= (\lambda_j - X_{11})(u_{1j} u_{2k} - u_{1k} u_{2j}) + (\lambda_k - X_{22})(u_{1j} u_{2k} - u_{1k} u_{2j})$$

$$= (\lambda_j + \lambda_k - X_{11} - X_{22})(u_{1j} u_{2k} - u_{1k} u_{2j})$$

Or rather

(156.1) $$u_{12,jk} = u_{1j}\, u_{2k} - u_{1k}\, u_{2j} ,$$

(156.2) $$\frac{d}{dt} u_{12,jk} = (\lambda_j + \lambda_k - X_{11} - X_{22})\, u_{12,jk} .$$

Now observe that, using the orthonormality of $\{u_{1j}\}$ and $\{u_{2k}\}$

$$\sum_{1 \le i < l \le N} (\lambda_i + \lambda_l)\, |u_{12,il}|^2$$

$$= \frac{1}{2} \sum_{1 \le i, l \le N} (\lambda_i + \lambda_l)\, |u_{12,il}|^2$$

$$= \frac{1}{2} \sum_{1 \le i, l \le N} (\lambda_i + \lambda_l)\, (u_{1i} u_{2l} - u_{1l} u_{2i})(\overline{u_{1i}}\, \overline{u_{2l}} - \overline{u_{1l}}\, \overline{u_{2i}})$$

$$= \frac{1}{2} \Big(\sum_{1 \le i, l \le N} (\lambda_i + \lambda_l)\, |u_{1i}|^2 |u_{2l}|^2$$

$$+ \sum_{1 \le i, l \le N} (\lambda_i + \lambda_l)\, |u_{1l}|^2 |u_{2i}|^2$$

$$- \sum_{1 \le i, l \le N} (\lambda_i + \lambda_l)\, u_{1i}\, \overline{u_{2i}}\, u_{2l}\, \overline{u_{1l}}$$

$$- \sum_{1 \le i, l \le N} (\lambda_i + \lambda_i)\, u_{1l}\, \overline{u_{2l}}\, u_{2i}\, \overline{u_{1i}} \Big)$$

$$= \sum \lambda_i |u_{1i}|^2 + \sum \lambda_l |u_{2l}|^2 + 0 + 0$$

$$= X_{11} + X_{22}$$

Thus we have for $1 \le j < k \le N$

(156.3) $$\frac{d}{dt} u_{12,jk} = \Big(\lambda_j + \lambda_k - \Big(\sum_{1 \le i < l \le N} \lambda_i + \lambda_l\, |u_{12,il}|^2 \Big) \Big)\, u_{12,jk}$$

It follows directly that the same calculation that produced (142.1) from (141.2) yields

Theorem 157.1

Let $X(t)$ solve the extended Toda flow (137.3). Let

(157.2) $$u_{12,jk}(t) = u_{1j}(t)\,u_{2k}(t) - u_{1k}(t)\,u_{2j}(t), \quad 1 \le j < k \le$$

Then

(157.3) $$u_{12,jk}(t) = \frac{u_{12,jk}(0)\, e^{(\lambda_j+\lambda_k)t}}{\left(\sum_{1\le i<\ell\le N} |u_{12,i\ell}(0)|^2 e^{2(\lambda_i+\lambda_\ell)t}\right)^{\frac{1}{2}}}. \quad \square$$

What is going on here? Consider the space of skew 2-tensors $\Lambda_2(\mathbb{C}^N)$ which is spanned by the lexicographically ordered basis $e_1\wedge e_2, e_1\wedge e_3, \ldots, e_1\wedge e_N, e_2\wedge e_3, \ldots, e_{N-1}\wedge e_N$

(157.4) ie $e_{i_1}\wedge e_{i_2} < e_{j_1}\wedge e_{j_2}$ if $i_1 < j_1$ or if $i_1 = j_1$ then $i_2 < j_2$,

of dimension $N(N-1)/2$. $\Lambda_2(\mathbb{C}^N)$ carries a natural inner product, generated by

$$(u\wedge v, \tilde u \wedge \tilde v) = \det\begin{pmatrix} (u,\tilde u) & (u,\tilde v) \\ (v,\tilde u) & (v,\tilde v)\end{pmatrix}$$

The operator $\Lambda(X) = X\otimes I + I \otimes X$

acts naturally in $\Lambda_2(\mathbb{C}^N)$,

$$\Lambda(X)\, u\wedge v = Xu\wedge v + u\wedge Xv$$

and if $X = X^*$ in $\Lambda(X)$ is self-adjoint in $\Lambda_2(\mathbb{C}^N)$

Now if $\{u_i\}$ are a complete orthonormal basis of eigenvectors for X, $Xu_i = \lambda_i u_i$, then $\{u_i\wedge u_j,\ i<j\}$ is a complete orthonormal basis of eigenvectors for $\Lambda(X)$ in $\Lambda_2(\mathbb{C}^N)$ with associated eigenvalues $\lambda_i + \lambda_j$ respectively,

$$\begin{aligned}\Lambda(X)(u_i\wedge u_j) &= Xu_i\wedge u_j + u_i\wedge Xu_j\\ &= \lambda_i(u_i\wedge u_j) + \lambda_j\, u_i\wedge u_j\\ &= \lambda_i+\lambda_j\ \ u_i\wedge u_j\end{aligned}$$

Now the first component of $u_i\wedge u_j$ in the ordered basis $(e_i\wedge e_j)$ is $(e_1\wedge e_2,\ u_i\wedge u_j)$

$$= \det\begin{pmatrix}(e_1, u_i) & (e_1, u_j)\\ (e_2, u_i) & (e_2, u_j)\end{pmatrix}$$

$$= u_{1i}\,u_{2j} - u_{1j}\,u_{2i}$$

which is exactly $u_{12,ij}$!

Also (cf (142.2))

$$(e_1 \wedge e_2, \; \Lambda_2(X)\, e_1 \wedge e_2)$$

$$= \sum_{1 \le i < l \le N} (\lambda_i + \lambda_l)\, |u_{12,il}|^2$$

Thus (156.3) can be written in the form

(159.1) $$\frac{d}{dt} u_{12,jk} = \Big((\lambda_j + \lambda_k) - (e_1 \wedge e_2, \; \Lambda_2(X)\, e_1 \wedge e_2) \Big) u_{12,jk}$$

which indicates that

if X solves Toda in $\mathbb{C}^N$ then $\Lambda_2(X) = X \otimes I + I \otimes X$ solves Toda in $\Lambda_2(\mathbb{C}^N)$

To see that this is indeed true, we utilize

Theorem 45.1. Note first that

(159.2) $$e^{t\Lambda_2(X)} = e^{tX} \otimes e^{tX}$$

Indeed $\frac{d}{dt} e^{t\Lambda_2(X)} = \Lambda_2(X)\, e^{t\Lambda_2 X} = (X \otimes I + I \otimes X)\, e^{t\Lambda_2(X)}$

On the other hand

$$\frac{d}{dt} e^{tX_0} \otimes e^{tX_0} = X_0 e^{tX_0} \otimes e^{tX_0} + e^{tX_0} \otimes X_0 e^{tX_0}$$

$$= (X_0 \otimes I + I \otimes X_0)\, e^{tX_0} \otimes e^{tX_0}$$

and so the LHS and the RHS of (154.3) satisfy the same differential equation. As $e^{t\Lambda_2(X_0)}\big|_{t=0} = e^{tX_0} \otimes e^{tX_0}\big|_{t=0}$

$= I \otimes I$, (159.3) follows.

Let $e^{tX_0} = Q(t)R(t)$ be the QR-factorization of e^{tX_0}, $Q(t)Q(t)^* = I$ and $R(t)$ is upper triangular with $R_{ii} > 0$.

Then

(160.1) $$e^{t\Lambda_2(X_0)} = QR \otimes QR = (Q \otimes Q)(R \otimes R)$$

Clearly $Q \otimes Q$ is unitary in $\Lambda_2(\mathbb{C}^N)$. On the other hand, suppose $e_{i_1} \wedge e_{i_2} > e_{j_1} \wedge e_{j_2}$. Then

$$(e_{i_1} \wedge e_{i_2},\ R \otimes R\, e_{j_1} \wedge e_{j_2})$$

$$= (e_{i_1} \wedge e_{i_2},\ Re_{j_1} \wedge Re_{j_2})$$

$$= \det \begin{pmatrix} (e_{i_1}, Re_{j_1}) & (e_{i_1}, Re_{j_2}) \\ (e_{i_2}, Re_{j_1}) & (e_{i_2}, Re_{j_2}) \end{pmatrix}$$

Suppose $i_1 > j_1$. Then $(e_{i_1}, Re_{j_1}) = 0$, and as $i_2 > i_1 > j_1$, we also have $(e_{i_2}, Re_{j_1}) = 0$. Thus $(e_{i_1} \wedge e_{i_2},\ R \otimes R\, e_{j_1} \wedge e_{j_2}) = 0$. If $i_1 = j_1$, then $i_2 > j_2$.

Then $(e_{i_2}, Re_{j_1}) = (e_{i_2}, Re_{i_1}) = 0$ as $i_2 > i_1$, and also $(e_{i_2}, R e_{j_2}) = 0$ as $i_2 > j_2$. Again we have $(e_{i_1} \wedge e_{i_2}, R \otimes R \; e_{j_1} \wedge e_{i_2}) = 0$. Thus $R \otimes R$ is upper triangular in the lexicographic ordering (157.4). Moreover,

$$(e_{i_1} \wedge e_{i_2}, \; R \otimes R \; e_{i_1} \wedge e_{i_2})$$

$$= \det \begin{pmatrix} (e_{i_1}, X e_{i_1}) & (e_{i_1}, X e_{i_2}) \\ (e_{i_2}, X e_{i_1}) & (e_{i_2}, X e_{i_2}) \end{pmatrix}$$

$$= X_{i_1 i_1} \, X_{i_2 i_2} > 0 \, ,$$

as $(e_{i_2}, X e_{i_1}) = 0$.

It follows in particular that (160.1) is the QR-factorization of $e^{t(X_0 \otimes I + I_0 \otimes X)}$, and hence by Theorem 145.1

$$X_2(t) = Q(t) \otimes Q(t) \, (X_0 \otimes I + I \times X_0) \, Q(t) \otimes Q(t)$$

solves the Toda flow in $\Lambda_2(\mathbb{C})$

(161.2) $$\frac{d}{dt} X_2(t) = [X_2, \tilde{B}(X_2)]$$

where $\tilde{B}(X_2) = X_{2-} - X_{2-}^*$, in the lexicographic ordering.

But

$$X_2(t) = Q(t)X_0Q(t)^T \otimes I + I \otimes Q(t)X_0Q(t)^T$$

$$= X(t) \otimes I + I \otimes X(t)$$

where $X(t)$ solves the Toda flow in $\mathbb{C}^N$. We have proved the following

Theorem 162.1

If $X(t)$ solves Toda in $\mathbb{C}^N$, then $X(t)\otimes I + I \otimes X(t)$ solves Toda in $\Lambda_2(\mathbb{C}^N)$.

This result explains (159.1). We have also proved the following ordering result.

Theorem 162.2

Let $A_N^{(2)} = \{X \in \Gamma_N :$ if $u = (u_1, \ldots, u_N)^T$ is an eigenvector of X ~~for~~ some eigenvalue λ, then $u_1 \neq 0$, and if $u \wedge v = (\langle e_1 \wedge e_2, u \wedge v\rangle, \ldots, \langle e_{N-1} \wedge e_N, u \wedge v\rangle)^T$ is an eigenvector of $\Lambda_2(X) = X \otimes I + I \otimes X$ for some eigenvalue $\lambda + \mu$, then $\langle e_1 \wedge e_2, u \wedge v\rangle \neq 0$.

Suppose $X(t)$ solves Toda with $X(0) = X_0 \in A_N^{(2)}$. Then as $t \to \infty$

$$X_{11}(t) \to \lambda_1, \quad X_{22}(t) \to \lambda_2.$$

For $X \in \Lambda_N^{(2)} \subset \Lambda_N$, we necessarily have $\lambda_1 > \lambda_2 > \cdots > \lambda_N$.

Proof: We have already shown, using (163.2) et seq, that $X_{11}(t) \to \lambda_1$. But from (159.1) we conclude, as in $\mathbb{C}^N$-case, that

(163.1)
$$u_{12,jk}(t) = \frac{u_{12,jk}(0)\, e^{t(\lambda_j+\lambda_k)}}{\left(\sum_{1\le i<l\le N} |u_{12,il}(0)|^2 e^{2t(\lambda_i+\lambda_l)}\right)^{1/2}}, \quad t \ge 0.$$

from it follows, again as in the $\mathbb{C}^N$ case, that

$$(e_1\wedge e_2,\ \Lambda_2(X(t))e_1\wedge e_2) \to \lambda_1+\lambda_2$$

as $\lambda_1+\lambda_2 > \lambda_i+\lambda_j$ for all $(i,j)\neq(1,2)$.

That is, $X_{11}(t) + X_{22}(t) = (e_1\wedge e_2,\ \Lambda_2(X(t))\, e_1\wedge e_2) \to \lambda_1+\lambda_2$,

from which we conclude that $X_{22}(t) \to \lambda_2$. $\square$.

Exercise 163.2

Show that $A_N^{(2)}$ is an open dense set of full measure in $\mathbb{R}^{N^2}$.

Exercise 163.c

Generalize Theorem 162.2, to give a condition that the Toda flow is ordering on an open dense set of full measure in $\mathbb{R}^{N^2}$