(loc. cit), which he derived using a very different, indirect argument.

Lecture 7

Exercise 112.1

Use (100.1) and (107.3) to show that

(112.2) $$x_j(t) = x_1(t) + \log \frac{\Delta_{j-2}(t)}{\Delta_{j-1}(t)} - (j-1)\log 4$$

where

(112.3) $$x_1(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(0) - \frac{2t}{N}\sum_{i=1}^{N}\lambda_i + (N-2)\log 2 + \frac{1}{N}\log \Delta_{N-1}(t)$$

$$= \frac{1}{N}\sum_{i=1}^{N} x_i(0) - \frac{2t}{N}\sum_{i=1}^{N}\lambda_i + (N-2)\log 2$$

$$+ \frac{1}{N}\log\left[\left(\prod_{i=1}^{N} u_i^2(1)\right) V^2(\lambda)\, \frac{e^{2\sum_i \lambda_i t}}{\sum_{i=1}^{N} e^{2\lambda_i t} u_i^2(1)}\right]$$

Use (112.2) (112.3) to rederive (109.3) and (109.4).

We now show that the Toda lattice is integrable in the sense of Liouville i.e. $H_T(x,y) = \frac{1}{2}\sum_{i=1}^{N} y_i^2 + \frac{1}{2}\sum_{i=1}^{N-1} e^{x_i - x_{i+1}}$ has N independent, Poisson commuting integrals $\{I_j\}$ on $(\mathbb{R}^{2N}, \omega = \sum_{i=1}^{N} dx_i \wedge dy_i)$

We take $I_j := \lambda_j$ = eigenvalues of X_0, $1 \le j \le N$.

We already know these are integrals for the Toda flow: it remains to show that they are independent and Poisson commute.

Theorem 113.1 Let $\{\lambda_j\}_{j=1}^N$ be the eigenvalues of an $N\times N$ Jacobi matrix X. Then

(113.2) $$\{\lambda_k, \lambda_j\} = 0 \qquad , \quad 1 \le k, j \le N$$

Proof: Write

$$X = \begin{pmatrix} a_1 & b_1 & & & 0 \\ b_1 & a_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_{N-1} \\ 0 & & & b_{N-1} & a_N \end{pmatrix}$$

Then if $X(\varepsilon)$, $X(0) = X$ is any smooth perturbation of X, $X(\varepsilon) \in J_N$, we have from $(X(\varepsilon) - \lambda_j(\varepsilon))u_j(\varepsilon) = 0$,

$$(\dot{X}(0) - \dot{\lambda}_j(0))u_j(0) + (X(0) + \lambda_j(0))\dot{u}_j(0) = 0 ,$$

$$(u_j(0), (\dot{X}(0) - \dot{\lambda}_j(0))u_j(0))$$

$$= -(u_j(0), (X(0) + \lambda_j(0))\dot{u}_j(0))$$

$$= -((X(0)u_j(0), \dot{u}_j(0)) - \lambda_j(0)\,u_j(0), \dot{u}_j(0))$$

$$= -2\lambda_j(0) \quad (u_j(0), \dot{u}_j(0))$$

But as the eigenvalues of $X(\varepsilon)$ are simple, we can, by the results on perturbation theory noted earlier (see lecture 5), assume that $u(\varepsilon)$ is (smooth) and normalized, ie $(u(\varepsilon), u(\varepsilon)) = 1$, which implies $(u(0), \dot{u}(0)) = 0$. Hence we conclude that

(114.1) $$\dot{\lambda}_j(0) = (u_j(0), \dot{X}(0)\, u_j(0))$$

Let E_{qp} denote the $N \times N$ matrix with 1 in the qp position and zero elsewhere, $1 \le q, p \le N$. That is,

(114.2) $$E_{qp}(i,j) = \delta_{iq}\, \delta_{jq} \qquad 1 \le i, j \le N.$$

From (114.1), we obtain, in particular

(114.3) $$\frac{\partial \lambda_i}{\partial a_\ell} = (u_i, E_{\ell\ell}\, u_i) = u_i^2(\ell), \qquad 1 \le \ell \le N$$

and

$$\frac{\partial \lambda_i}{\partial b_\ell} = (u_i, (E_{\ell\ell+1} + E_{\ell+1\ell})\, u_i) = 2\, u_i(\ell)\, u_i(\ell+1), \qquad 1 \le \ell \le N-1.$$

Hence

$$(115.1) \qquad \frac{\partial \lambda_j}{\partial y_i} = \frac{1}{(-2)} \frac{\partial \lambda_j}{\partial a_i} = -\frac{1}{2} u_j^2(i) \qquad 1 \le i \le N$$

and

$$(115.2) \qquad \frac{\partial \lambda_j}{\partial x_i} = \frac{\partial \lambda_j}{\partial b_i} \frac{\partial b_i}{\partial x_i} + \frac{\partial \lambda_j}{\partial b_{i-1}} \frac{\partial b_{i-1}}{\partial x_i}$$

$$= \frac{\partial \lambda_j}{\partial b_i} \frac{1}{2} b_i - \frac{\partial \lambda_j}{\partial b_{i-1}} \frac{1}{2} b_{i-1}$$

$$= u_j(i) u_j(i+1) b_i - u_j(i-1) u_j(i) b_{i-1}$$

$$= u_j(i) \left(u_j(i+1) b_i - u_j(i-1) b_{i-1} \right), \quad 1 \le i \le N$$

where

$$b_0 = b_N \equiv 0$$

<u>Claim</u>: For $1 \le m \le N-1$

$$(115.3) \qquad (\lambda_j - \lambda_k) \sum_{i=1}^{m} u_k(i) u_j(i) = b_m \left(u_k(m) u_j(m+1) - u_k(m+1) u_j(m) \right)$$

$$\equiv a_{kj}(m)$$

Indeed

$$\lambda_j \sum_{i=1}^{m} u_k(i) u_j(i) = \sum_{i=1}^{m} u_k(i) (X u_j)(i)$$

$$= \sum_{i=1}^{m} u_k(i) \left(b_i u_j(i+1) + a_i u_j(i) + b_{i-1} u_j(i-1) \right)$$

$$= \sum_{i=2}^{m+1} u_k(i-1) b_{i-1} u_j(i) + \sum_{i=1}^{n} u_k(i) a_i u_j(i)$$

$$+ \sum_{i=0}^{m-1} u_k(i+1) b_i u_j(i)$$

$$= \sum_{i=1}^{m} \left(b_{i-1} u_k(i-1) + a_i u_k(i) + b_i u_k(i+1) \right) u_j(i)$$

$$+ b_m u_k(m) u_j(m+1) - \cancel{b_0 u_k(0) u_j(1)}$$

$$- b_m u_k(m+1) u_j(m) + \cancel{u_k(1) u_j(1)} b_0$$

$$= b_m \left(u_k(m) u_j(m+1) - u_k(m+1) u_j(m) \right)$$

which proves the Claim. Note that (115.3) is just summing by parts.

Now from (115.1) (115.2)

$$\{\lambda_k, \lambda_j\} = \sum_{i=1}^{N} \left[\frac{\partial \lambda_k}{\partial x_i} \frac{\partial \lambda_j}{\partial y_i} - \frac{\partial \lambda_k}{\partial y_i} \frac{\partial \lambda_j}{\partial x_i} \right] \tag{116.1}$$

$$= \sum_{i=1}^{N} \Big[u_k(i) \left(u_k(i+1) b_i - u_k(i-1) b_{i-1} \right) \left(-\tfrac{1}{2}\right) u_j^2(i)$$

$$- \left(-\tfrac{1}{2}\right) u_k^2(i)\, u_j(i) \left(u_j(i+1) b_i - u_j(i-1) b_{i-1} \right)$$

$$= \frac{1}{2} \sum_{i=1}^{N} u_k(i) u_j(i) \Big[u_k(i) \left(u_j(i+1) b_i - u_j(i-1) b_{i-1} \right]$$
$$- u_j(i) \left(u_k(i+1) b_i - u_k(i-1) b_{i-1} \right)$$

$$= \frac{1}{2} \sum_{i=1}^{N} u_k(i) u_j(i) \Big\{ b_i \left(u_k(i) u_j(i+1) - u_k(i+1) u_j(i) \right)$$
$$+ b_{i-1} \left(u_k(i-1) u_j(i) - u_k(i) u_j(i-1) \right)$$

Now from (115.3)

$$(\lambda_j - \lambda_k)\, u_k(i)\, u_j(i) = b_i \left(u_k(i) u_j(i+1) - u_k(i+1) u_j(i) \right) - b_{i-1} \left(u_k(i-1) u_j(i) - u_k(i) u_j(i-1) \right)$$

$$= a_{kj}(i) - a_{kj}(i-1)$$

Thus

$$a_{kj}^2(i) - a_{kj}^2(i-1) = \left(a_{kj}(i) - a_{kj}(i-1) \right)\left(a_{kj}(i) + a_{kj}(i-1) \right)$$

$$= (\lambda_j - \lambda_k)\, u_k(i) u_j(i) \left(a_{kj}(i) + a_{kj}(i-1) \right)$$

It follows from (116.1) that

$$\{\lambda_k, \lambda_j\} = \frac{1}{2} \frac{1}{\lambda_j - \lambda_k} \sum_{i=1}^{N} a_{kj}^2(i) - a_{kj}^2(i-1)$$

$$= \frac{1}{2} \frac{1}{\lambda_j - \lambda_k} \left(a_{kj}^2(N) - a_{kj}^2(0) \right)$$

$$= 0$$

as $b_N = b_0 = 0$.

This completes the proof of (113.2). $\square$

<u>Theorem 117.1</u>

Let $\{\lambda_j\}_{j=1}^{N}$ be the eigenvalues of an $N \times N$ Jacobi matrix X. Then $\lambda_1, \ldots, \lambda_N$ are independent.

Proof: We must show that if

(117.2) $\sum_{k=1}^{N} \gamma_k \, d\lambda_k = 0$

for some γ_i, $i=1,\dots,N$, then $\gamma_1=\gamma_2=\dots=\gamma_N=0$.

From (117.2), we have

(118.1) $$0=\sum_{k=1}^{N}\gamma_k\frac{\partial\lambda_k}{\partial x_i}=\sum_{k=1}^{N}\gamma_k u_k(i)\left(u_k(i+1)b_i-u_k(i-1)b_{i-1}\right),\quad 1\le i\le N$$

and

(118.2) $$0=\sum_{k=1}^{N}\gamma_k\frac{\partial\lambda_k}{\partial y_i}=-\frac{1}{2}\sum_{k=1}^{N}\gamma_k u_k^2(i),\qquad i=1,\dots,N$$

From (118.1), we obtain

$$L_i\equiv\sum_{k=1}^{N}\gamma_k u_k(i+1)u_k(i)b_i=\sum_{k=1}^{N}\gamma_k u_k(i)u_k(i-1)b_{i-1}\equiv L_{i-1}$$

But $L_0=0$ and so $L_i=0$ for $1\le i\le N$

L_N is trivially equal to zero as $b_N=0$, but $b_i>0$ for $1\le i\le N-1$

and so we conclude

(118.3) $$\sum_{k=1}^{N}\gamma_k u_k(i)u_k(i+1)=0,\qquad 1\le i\le N-1.$$

From (118.2), we have

(118.4) $$\sum_{k=1}^{N}\gamma_k u_k^2(i)=0,\qquad 1\le i\le N$$

Now for $1\le i\le N$, we have from (118.3) and $Xu_k=\lambda_k u$

$$0=b_i\sum_{k=1}^{N}\gamma_k u_k(i)u_k(i+1)$$

$$= \sum_{k=1}^{N} \gamma_k\, u_k(i) \left((\lambda_k - a_i) u_k(i) - b_{i-1} u_k(i-1) \right)$$

$$= \sum_{k=1}^{N} \gamma_k \lambda_k u_k^2(i) \quad , \text{so}$$

(119.0) $$\sum_{k=1}^{N} \gamma_k \lambda_k u_k^2(i) = 0 \quad , \quad 1 \le i \le N$$

Also, for $1 \le i \le N-1$

(119.1) $$\sum_{k=1}^{N} \gamma_k \lambda_k u_k(i) u_k(i+1) = \sum_{k=1}^{N} \gamma_k (\lambda_k - a_i) u_k(i)\, u_k(i+1)$$

$$= \sum_{k=1}^{N} \gamma_k \left(b_i u_k(i+1) + b_{i-1} u_k(i-1) \right) u_k(i+1)$$

$$= b_{i-1} \sum_{k=1}^{N} \gamma_k\, u_k(i-1)\, u_k(i+1)$$

On the other hand,

$$\sum_{k=1}^{N} \gamma_k \lambda_k u_k(i)\, u_k(i+1) = \sum_{k=1}^{N} \gamma_k\, u_k(i) \left(\lambda_k - a_{i+1} \right) u_k(i+1)$$

$$= \sum_{k=1}^{N} \gamma_k\, u_k(i) \left(b_{i+1} u_k(i+2) + b_i u_k(i) \right)$$

$$= b_{i+1} \sum_{k=1}^{N} \gamma_k\, u_k(i)\, u_k(i+2)$$

Thus for $1 \le i \le N-1$

$$b_{i+1} \sum_{k=1}^{N} \gamma_k\, u_k(i)\, u_k(i+2) = b_{i-1} \sum_{k=1}^{N} \gamma_k\, u_k(i-1)\, u_k(i+1)$$

Now for $i = 1$, the RHS $= 0$, and so as $b_1 \ne 0$ we have from the RHS

$$\sum_{k=1}^{N} \gamma_k\, u_k(1)\, u_k(3) = 0$$

Now for $i = 2$, the RHS $= 0$, and as $b_3 \ne 0$, we conclude that

$$\sum_{k=1}^{N} \gamma_k \, u_k(i) \, u_k(4) = 0$$

Continuing we obtain

$$\sum_{k=1}^{N} \gamma_k \, u_k(i) \, u_k(i+2) = 0 \qquad i=1,\ldots,N-2 \tag{120.1}$$

Inserting this relation into (118.1), we obtain

$$\sum_{k=1}^{N} \gamma_k \lambda_k \, u_k(i) \, u_k(i+1) = 0 \qquad , \; 1 \le i \le N-1. \tag{120.2}$$

Thus we see that (118.3)(118.4) imply (119.0)(120.2).

We can clearly repeat the above calculation and conclude that for $0 \le \ell \le N-1$

$$\sum_{k=1}^{N} \gamma_k \lambda_k^{\ell} \, u_k^2(i) = 0 \qquad , \; i=1,\ldots,N \tag{120.3}$$

and

$$\sum_{k=1}^{N} \gamma_k \lambda_k^{\ell} \, u_k(i) \, u_k(i+1) = 0 \qquad , \; 1 \le i \le N-1 \tag{120.4}$$

But $\{\lambda_k^{\ell}\}_{1 \le k, \ell \le N}$ is a van der Monde matrix and so for any $i \in \{1,\ldots,N\}$, we must have

$$\gamma_k \, u_k^2(i) = 0 \qquad , \; k=1,\ldots,N.$$

But $\sum_{i=1}^{N} u_k^2(i) = 1$ and so we have $\gamma_k = 0$, $k=1,\ldots,N$. $\square$