Lecture 9 We now show that Moser's argument on p90 et seq on the long-time behavior of solutions of the Toda flow applies in the full symmetric matrix case. In place of (10.1), $\frac{dX}{dt} = [X, B(X)]$, $X(0) \in \Sigma_N$, we will consider more generally its complexification

(135.1) $$\frac{dX}{dt} = [X, \tilde{B}(X)], \quad X(0) = X_0 = X_0^* \in T_N$$

where T_N denotes the $N \times N$ Hermitian matrices and

(135.2) $$\tilde{B}(X) = X_- - X_-^* = -(\tilde{B}(X))^*$$

Using suitable analogs of the calculations on p10 et seq it is easy to show that (135.1) has a unique global solution $X(t) = X(t)^*$ with spec $(X(t))$ = spec X_0. Clearly if $X_0 = \bar{X}_0 = X_0^T$, then (135.1) reduces to (10.1).

Theorem (135.3)

Let $X(t)$, $t \geq 0$, solve (135.1) with $X(0) = X_0 = X_0^*$.

Then as $t \to \infty$,

$$X(t) \to \operatorname{diag}(\lambda_{\pi(1)}, \dots, \lambda_{\pi(N)})$$

where $\lambda_{\pi(1)}, \dots, \lambda_{\pi(N)}$, is some permutation of the eigenvalues of X_0.

<u>Proof:</u> We have $\frac{dX}{dt} = X\hat{B} + (X\tilde{B})^*$, and using the complex inner product for $\mathbb{C}^N$, we obtain

$$\frac{dX_{11}}{dt} = 2\mathrm{Re}\,((e_1, X\tilde{B}e_1))$$

$$= 2\mathrm{Re}\,(Xe_1, (X - X_{11})e_1)$$

$$= 2\sum_{i=1}^{N} |X_{i1}|^2 - 2|X_{11}|^2$$

$$= 2\sum_{i=2}^{N} |X_{i1}|^2$$

Following Moser's argument we now conclude that

(136.1) $$X_{11}(t) \uparrow X_{11}(\infty) \quad \text{and} \quad \sum_{i=2}^{N} |X_{i1}(t)|^2 \to 0.$$

Now

$$\frac{d(X_{11} + X_{22})}{dt} = 2\mathrm{Re}\,\left((e_1, X\tilde{B}e_1) + (e_2, X\hat{B}e_2)\right)$$

$$= 2\mathrm{Re}\,\left(\sum_{i=2}^{N} |X_{i1}|^2 + \sum_{i=1}^{N} |X_{i2}|^2 - |X_{22}|^2 - 2|X_{12}|^2\right)$$

$$= 2 \sum_{i=1}^{2} \sum_{2<j\le N} |X_{ji}(t)|^2$$

and we conclude that as $t \to \infty$

$$X_{22}(t) \to X_{22}(\infty) \quad \text{and} \quad \sum_{i=3}^{N} |X_{i2}(t)|^2 \to 0$$

Continuing we find for $1 \le k \le N$

$$\frac{d(X_{11} + \cdots + X_{kk})}{dt} = 2 \sum_{i=1}^{k} \sum_{k<j\le N} |X_{ij}(t)|^2$$

which leads to the desired result. $\square$.

Remark 137.1

Note that, unlike the Jacobi case, $X(t)$ is not always ordering i.e.

$\lambda_{\pi(1)}, \ldots, \lambda_{\pi(N)}$ is not always ordered. Indeed if

X_0 is diagonal, then $X(t) = X_0$ for all $t \ge 0$, and

so no re-ordering can take place. Below we will

discuss a sufficient condition that $X(t) \in \Sigma_N$ is ordering.

Remark 137.2

The extended generalized flow, $p \ge 3$,

(137.3) $$\frac{dX}{dt} = [X, \tilde{B}(X^{p-1})], \quad X(0) = X_0 = X_0^*$$

is not always diagonalizing. For example, in the case

$p=3$ with $X_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X_0^* = X_0^T$. Then by (131.1), $X(t) = Q(t)^T X_0 Q(t)$ and

$X(t)^{p-1} = X(t)^2 = Q(t)^T X_0^2 Q(t) = I$ as $X_0^2 = I$.

In particular $\tilde{B}(X^{p-1}) = 0$ and so $X(t) = \text{const} = X_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

However, it follows as in Lecture 1, that if $X(t)$

solves (137.3) for any $p \geq 2$, then $\frac{dX^{p-1}}{dt} = [X^{p-1}, \tilde{B}(X^{p-1})]$,

which is the Toda flow. Hence $X^{p-1}(t)$ converges to the

diagonal matrix $\text{diag}(\lambda_{\pi(1)}^{p-1}, \ldots, \lambda_{\pi(N)}^{p-1})$ as $t \to \infty$.

But, as above, $X(t)$ may not converge to a diagonal

matrix

<u>Exercise</u> Does $X(t)$ always converge to some matrix under (137.3)?

We now prove the analog of (84.1) (84.2) for the extended generalized flow (137.3). This requires some interpretation as the eigenvalues of a general Hermitian matrix X may not be simple and, in particular, the eigenvalues and eigenvectors may not be smooth function of X. We proceed as follows (see D+Trogdon, Univ. for...). For $X(0) = X_0$ there exists a (not necessarily unique) unitary matrix U_0 such that $X_0 = U_0 \Lambda U_0^*$ where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$.

Let $Q(t)$ solve (cf (13.11))

(138.1) $$\frac{dQ}{dt} = Q\,\tilde{B}(X^{p-1}(t)), \qquad Q(0) = I,$$

where $X(t)$ solves (137.3). Then

$$\frac{d}{dt} QQ^* = \dot{Q}Q^* + Q\dot{Q}^* = Q\tilde{B}Q^* + Q\tilde{B}^*Q^* = 0$$

as $\tilde{B} = -\tilde{B}^*$. Thus $Q(t)$ is unitary, $Q(t)\,Q(t)^* = \text{const.} = I$

Following (14.1) we find

(139.1) $$X(t) = Q(t)^* X_0 Q(t)$$

and

(139.2) $$X(t) = U(t) \Lambda U(t)^*$$

where $U(t)$ is the unitary matrix

(139.3) $$U(t) = Q(t)^* U_0 .$$

Then (as $Q(t)$ is smooth,) the j^{th} column $u_j(t)$ of $U(t)$ is a smooth eigenvector of $X(t)$ corresponding to the (constant) eigenvalue λ_j. From the eigenvalue equation $(X(t) - \lambda_j) u_j(t) = 0$ we obtain

$$\dot{X}(t) u_j(t) + (X(t) - \lambda_j) \dot{u}_j(t) = 0$$

which yields using (137.3)

$$(X(t) - \lambda_j)(\dot{u}_j(t) + \tilde{B}(X^{p-1}(t)) u_j(t)) = 0$$

This last equation implies that $\dot{u}_j(t) + B(X^{p-1}(t)) u_j(t)$ must be a (possibly time dependent) linear combination of the eigenvectors corresponding to λ_j. Let

$$U_j(t) = [u_{j1}(t), \dots, u_{jm}(t)]$$

be the eigenvectors corresponding to a repeated eigenvalue λ_j so that for $i = 1, .., m$

(140.1) $$\dot{u}_{j_i}(t) + \tilde{B}(X^{p-1}(t))\, u_{j_i}(t) = \sum_{k=1}^{m} d_{ki}(t)\, u_{j_k}(t)$$

and so

(140.2) $$\left(\frac{d}{dt} + \tilde{B}(X^{p-1}(t))\right) U_j(t) = U_j(t)\, D(t)$$

where $$D(t) = (d_{ki}(t))_{k,i=1}^{m}.$$

Clearly $D(t)$ is smooth.

Note that $U_j^*(t)\, U_j(t) = I_m$, the $m \times m$ identity matrix.

Then multiplying (140.1) on the left by $U_j^*(t)$ and

Then multiplying the conjugate transpose of (140.2) on the right by $U_j(t)$, we obtain

(140.3) $$U_j^*(t)\, \dot{U}_j(t) + U_j^*(t)\, \tilde{B}(X(t))\, U_j(t) = D(t)$$

and

(140.4) $$\dot{U}_j^*(t)\, U(t) + U_j^*(t)\, \tilde{B}(X(t))^*\, U_j(t) = D^*(t).$$

Because $\frac{d}{dt} U_j^*(t)\, U_j(t) = 0$ and $\tilde{B}(X^{p-1}(t))$ is skew Hermitian, the addition of these two equations yields

$D(t) = -D^*(t)$. Let $S(t)$ solve $\dot{S}(t) = -D(t)\,S(t)$, $S(0) = I_m$. Then one finds as above that $S^*(t)\,S(t) = \text{const.} = I_m$. In particular

$$\tilde{U}_j(t) = U_j(t)\,S(t)$$

has orthonormal columns and we find

(141.1) $$\left(\frac{d}{dt} + \tilde{B}(X^{p-1}(t))\right)\tilde{U}_j(t) = U_j(t)\,D(t)\,S(t) - U_j(t)\,D(t)\,S(t) = 0$$

We conclude that a smooth normalization for the eigenvectors of $X(t)$ can always be chosen so that $D(t) = 0$ in (140.2). So without loss of generality we can assume that $U(t)$ solves (140.2) with $D(t) = 0$.

Then for $U(t) = (u_{ij}(t))_{i,j=1}^{N}$

$$\dot{u}_{1j}(t) = -(e_1, \tilde{B}(X^{p-1}(t))\,u_j(t))$$

$$= (\tilde{B}(X^{p-1}(t))\,e_1, u_j(t))$$

$$= ((X^{p-1}(t) - (X^{p-1}(t))_{11})\,e_1, u_j(t))$$

(141.2) i.e. $$\dot{u}_{1j}(t) = (\lambda_j^{p-1} - (X^{p-1}(t))_{11})\,u_{1j}(t)$$

Using $(X^{p-1}(t))_{11} = \sum_{j=1}^{N} \lambda_j^{p-1} |U_{1j}(t)|^2$, a direct calculation shows the following

Theorem 142.0 Let $X(t)$ solve the generalized extended Toda flow ((137.3)).

(142.1) $$U_{1j}(t) = \frac{U_{1j}(0)\, e^{\lambda_j^{p-1} t}}{\left(\sum_{j=1}^{N} |U_{1j}(0)|^2 e^{2\lambda_j^{p-2} t} \right)^{\frac{1}{2}}}.$$

Then there exists a smooth orthonormal eigenbasis $U(t)$ for $X(t)$ such that

Moreover for $p=2$, in particular, we have

(142.2) $$X_{11}(t) = \sum_{j=1}^{N} \lambda_j |U_{1j}(t)|^2$$

and

(142.3) $$E(t) \equiv \sum_{k=2}^{N} |X_{1k}(t)|^2 = \sum_{j=1}^{N} (\lambda_j - X_{11}(t))^2 |U_{1j}(t)|^2$$

Proof: We must prove (142.3). From the spectral representation $X(t) = U(t) \Lambda U(t)^*$, we have

$$X_{1k}(t) = \sum_{j=1}^{N} \lambda_j U_{1j}^*(t) U_{kj}(t)$$

and hence

$$E(t) = \sum_{k=2}^{N} |X_{1k}(t)|^2 = \sum_{k=2}^{N} X_{1k}(t) X_{k1}(t) = (X^2(t))_{11} - X_{11}^2(t$$

$$= \sum_{k=1}^{N} \lambda_k^2 \, |U_{1k}(t)|^2 - \left(\sum_{k=1}^{N} \lambda_k |U_{1k}(t)|^2 \right)^2$$

$$= \sum_{k=1}^{N} |\lambda_k - X_{11}(t)|^2 \, |U_{1k}(t)|^2,$$

which proves (142.3). $\square$

As we will see $E(t)$ controls the computation of λ_1, the top eigenvalue of X_0.

Note that

(143.0) $$\lambda_1 - X_{11}(t) = \sum_{j=1}^{N} (\lambda_1 - \lambda_j) |U_{1j}(t)|^2 .$$

Suppose the eigenvalues of X_0 are arranged in non-increasing order

(143.1) $$\lambda_1 = \lambda_2 = \cdots = \lambda_m > \lambda_{m+1} \geq \lambda_{m+2} \geq \cdots \geq \lambda_N$$

Suppose $U_{1j}(0) \neq 0$ for some $1 \leq j \leq m$. Then without loss of generality, after permutation, we can assume that $U_{11}(t), \ldots, U_{1\ell}(t)$ are non-zero for some $1 \leq \ell \leq m$, and $U_{1\ell+1}(t), \ldots, U_{1m}(t) \equiv 0$. It now follows from (142.1) that

(143.2) $$U_{1j}(t) \to 0 \quad \text{for} \quad \ell+1 \leq j \leq N$$

and

(143.3) $$U_{1j}(t) \to U_{1j}(0) \Big/ \left(\sum_{i=1}^{\ell} |U_{1i}(0)|^2 \right)^{1/2} \quad \text{for} \quad 1 \leq j \leq \ell .$$

From (143.0) we see that

$$(144.1) \qquad \lambda_1 - X_{11}(t) = \sum_{j=m+1}^{N} (\lambda_1 - \lambda_j)\, |U_{1j}(t)|^2 .$$

and

$$(144.2) \qquad E(t) = \sum_{j=1}^{m} (\lambda_1 - X_{11}(t))^2 |U_{1j}(t)|^2 + \sum_{j=m+1}^{N} (\lambda_j - X_{11}(t)) |U_{1j}(t$$

go to zero as $t \to \infty$

While $\lambda_1 - X_{11}(t)$ is of course the true error in computing λ_1, we will use $E(t)$ to determine a convergence criterion as it is easily observable; Indeed, it follows from the min-max principle, that if $E(t) < \varepsilon$ then $|X_{11}(t) - \lambda_i| < \varepsilon$ for some i. With high probability (see ? below), $\lambda_i = \lambda_1$.

Now it turns out that there is a remarkable way to solve (10.1) or (135.1) (and the more general flows generated by ~~By~~ the Hamiltonians $H_p(X) = \frac{1}{p} \operatorname{tr} X^p$, explicitly. For more information see, e.g., [Symes] [DLNT] and the references therein.

Exercise 145.0 (QR factorization). Let Y be an invertible, complex matrix. Then Y has a unique factorization $Y = QR$ where Q is unitary, and R is upper triangular with $R_{ii} > 0$, $i = 1, \dots, N$. 145

Theorem 145.1

Let $X(t)$ solve the generalized and extended flow

(145.2) $$\frac{dX}{dt} = [X, \tilde{B}(X^{p-1})], \qquad X(0) = X_0 = X_0^*$$

Let

(145.3) $$e^{tX_0^{p-1}} = Q(t)\, R(t)$$

be the QR factorization for $e^{tX_0^{p-1}}$. Then

(145.4) $$X(t) = Q(t)\, X_0\, Q(t)^*$$

Proof: The QR factorization of a matrix Y is just the result of applying Gram-Schmidt to the columns of Y, starting from the left. It follows that as $e^{tX_0^{p-1}}$ is differentiable and non-singular for all t, that $Q(t)$, and $R(t)$

are differentiable functions of t. From (145.3)

we have

$$X_0^{p-1} QR = \dot{Q}R + Q\dot{R}$$

or

(146.1) $$\hat{X}(t)^p \equiv Q^* X_0^{p-1} Q = Q^*\dot{Q} + \dot{R}R^{-1}, \text{ where}$$

(146.2) $$\hat{X}(t) = Q^* X_0 Q$$

As $\dot{R}R^{-1}$ is upper triangular, it follows that

$$(Q^*\dot{Q})_- = \hat{X}_-$$

But $Q^*Q = I$ implies that $\dot{Q}^*Q + Q^*\dot{Q} = 0$

and hence $Q^*\dot{Q}$ is skew-adjoint with a purely imaginary diagonal i.e.

$$Q^*\dot{Q} = (Q^*\dot{Q})_- - (Q^*\dot{Q})_-^* + D$$

where $D + D^*$ is diagonal. But from (146.1), as $\hat{X}$ is Hermitian,

$$D = \operatorname{diag} \hat{X}^{p-1} + \operatorname{diag} \dot{R}R^{-1}$$

is real. Thus we must have $D = 0$ and hence

$$Q^*\dot{Q} = \tilde{B}(\hat{X}^{p-1})$$

or

(146.3) $$\frac{dQ}{dt} = Q\,\tilde{B}(\hat{X}^{p-1}), \quad Q(0) = I$$

Thus

$$\frac{d\hat{X}}{dt} = \dot{Q}^* X_0 Q + Q^* X_0 \dot{Q}$$

$$= -\tilde{B}(\hat{X}^{p-1})\hat{X} + \hat{X}\tilde{B}(\hat{X}^{p-1})$$

$$= [\hat{X}, \tilde{B}(\hat{X}^{p-1})].$$

where $\hat{X}(0) = X_0$. As the solution to (145.2) is unique. This completes the proof of (145.4). $\square$.