Scott, Foresman and Company, Glenview, Illinois London, 1971

Lecture 2

References:

- H. Goldstein, Classical Mechanics, Addison-Wesley 1950
- V. Arnold, Mathematical Methods of Classical Mechanics, Springer-Verlag, New York, 1978
- R. Abraham and J.E. Marsden, Foundations of Mechanics, Second Edition, Benjamin/Cummings, Reading, Massachusetts, 1978
- J. Moser and E. Zehnder, Notes on Dynamical Systems, Courant Lecture Notes 12, Amer. Math. Soc., Providence, 2005
- F. Warner, Foundations of Diff. Manifolds and Lie Groups,

The goal of the next few lectures is to give a rapid but elementary introduction to Hamiltonian mechanics, particularly integrable Hamiltonian systems, with a view to describing various results that we need in analyzing the Toda algorithm. The reader is of course encouraged to consult the References for more details.

The space $M = \mathbb{R}^{2n}$ is an example of a symplectic

<u>manifold</u> i.e., it is an

- even dimensional manifold, with a
- non-degenerate 2-form ω, i.e. $\omega(u,v)=0$ for $u \in T_m M$ and for all $v \in T_m M$, implies $u=0$, and
- which is closed, i.e. $d\omega = 0$

Non-degeneracy and skew symmetry, $\omega(u,v) = -\omega(v,u)$, imply that M must be even dimensional.

For $\mathbb{R}^{2n}$, the standard 2-form is $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$.

Clearly $d\omega = 0$. If $v = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i} \equiv \begin{pmatrix} a \\ b \end{pmatrix}$, and $v' = \begin{pmatrix} a' \\ b' \end{pmatrix}$, then a simple calculation shows that

$\omega(v, v') = \left(\begin{pmatrix} a \\ b \end{pmatrix}, J \begin{pmatrix} a' \\ b' \end{pmatrix} \right)$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, I is the $n \times n$ identity, and $(\cdot,\cdot)$ denotes the standard inner product on $\mathbb{R}^{2n}$. From this it is clear that

$$\omega(v, v') = 0 \text{ for all } v' \Rightarrow J^T \begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

i.e. $v = 0$, so ω is indeed non-degenerate.

Functions (i.e. 'Hamiltonians') $H : M \to \mathbb{R}$ generate vector fields through ω in the following way. At any point $m \in M$, dH_m is a 1-form: hence $v \mapsto dH_m(v)$ is a linear map from $T_m(M)$ to $\mathbb{R}$. Thus, as ω is non-degenerate, there exists a unique vector $v_H(m) \in T_m M$ such that

(28.1) $$dH(v) = \omega(v_H, v) \quad \text{for all } v \in T_m M.$$

In the case of $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i)$, equation (28.1) becomes for $v = \binom{a}{b}$

$$dH(v) = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial x_i} a_i + \frac{\partial H}{\partial y_i} b_i \right) = \left(\begin{pmatrix} H_x \\ H_y \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right) = \left(v_H, J \begin{pmatrix} a \\ b \end{pmatrix} \right).$$

Hence $\begin{pmatrix} H_x \\ H_y \end{pmatrix} = J^T v_H$, which implies as $J J^T = I_{2n}$,

$$v_H = J \begin{pmatrix} H_x \\ H_y \end{pmatrix} = \begin{pmatrix} H_y \\ -H_x \end{pmatrix}.$$

Thus H gives rise to the standard Hamiltonian vector field

(28.2) $$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = v_H = \begin{pmatrix} H_y \\ -H_x \end{pmatrix}$$

The fact that ω is non-degenerate plays an obvious role here. The role that ω is closed, $d\omega = 0$, plays s more subtle. If H, K are two Hamiltonians on M, we define their <u>Poisson bracket</u> $\{H, K\}$ through

(29.1) $$\{H, K\} = \omega(v_H, v_K)$$

Note that

(29.2) $$\{H, K\} = -\{K, H\}.$$

(29.3) and that, as ω is non-degenerate, $\{H, K\} = 0 \;\; \forall K \Rightarrow H = \text{constant}$

Clearly $\{H, K\} = dH(v_K) = v_K(H) = \frac{dH}{dt}$

where $\frac{dH}{dt}$ is the derivative of H in the direction of v_K.

Thus

(29.4 $$\frac{dH}{dt} = \{H, K\}$$

describes the changes of H along the flow generated by K.

In particular $\frac{dK}{dt} = \{K, K\} = 0$ as $\{\cdot, \cdot\}$ is skew.

Thus K is a conserved quantity for the flow it generates.

The Poisson bracket acts like a derivative and satisfies

Leibnitz's rule in the sense that

(30.1) $$\{HK, L\} = H\{K, L\} + \{H, L\}K$$

for all Hamiltonians H, K, L. Indeed

$$\{HK, L\} = \frac{d}{dt} HK = \frac{dH}{dt} K + H \frac{dK}{dt} = \{H, L\}K + H\{K, L\}$$

Observe that for $\left(\mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i\right)$

$$\{H, K\} = \omega(v_H, v_K) = \left(J\begin{pmatrix} H_x \\ H_y \end{pmatrix}, J\left(J\begin{pmatrix} K_x \\ K_y \end{pmatrix}\right)\right)$$

$$= \left(\begin{pmatrix} H_x \\ H_y \end{pmatrix}, J\begin{pmatrix} K_x \\ K_y \end{pmatrix}\right)$$

ie

(30.2) $$\{H, K\} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial x_i} \frac{\partial K}{\partial y_i} - \frac{\partial H}{\partial y_i} \frac{\partial K}{\partial x_i} \right)$$

Thus

$$\{\{H, K\}, L\} = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\{H, K\}\right) \frac{\partial L}{\partial y_i} - \left(\frac{\partial}{\partial y_i}\{H, K\}\right) \frac{\partial L}{\partial x_i}$$

$$= \sum_{i=1}^{n} \left\{ \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{n}\left(\frac{\partial H}{\partial x_k}\frac{\partial K}{\partial y_k} - \frac{\partial H}{\partial y_k}\frac{\partial K}{\partial x_k}\right)\right) \frac{\partial L}{\partial y_i} \right.$$

$$\left. - \frac{\partial}{\partial y_i}\left(\sum_{k=1}^{n}\left(\frac{\partial H}{\partial x_k}\frac{\partial K}{\partial y_k} - \frac{\partial H}{\partial y_k}\frac{\partial K}{\partial x_k}\right)\right) \frac{\partial L}{\partial x_i} \right.$$

$$= \sum_{i,k} \Big[L_{y_i}\left(H_{x_i x_k} K_{y_k} + H_{x_k} K_{y_k x_i} - H_{y_k x_i} K_{x_k} - H_{y_k} K_{x_i x_k}\right)$$

$$- L_{x_i}\left(H_{x_k y_i} K_{y_k} + H_{x_k} K_{y_i y_k} - H_{y_i y_k} K_{x_k} - H_{y_k} K_{x_k y_i}\right)$$

Thus $\{\{H,K\},L\}$

$$= \sum_{i,k} \Big[H_{x_i x_k} K_{y_k} L_{y_i} - \cancel{H_{y_k} K_{x_i x_k} L_{y_i}} + \cancel{H_{y_i y_k} K_{x_k} L_{x_i}}$$

$$- \cancel{H_{x_k} K_{y_i y_k} L_{x_i}}$$

$$- \cancel{H_{y_k x_i} K_{x_k}} L_{y_i} + H_{x_k} \cancel{K_{y_k x_i}} L_{y_i}$$

$$- \boxed{H_{x_k y_i} K_{y_k} L_{x_i}} + H_{y_k} K_{x_k y_i} L_{x_i} \Big].$$

By symmetry

$$\{\{K,L\},H\} = \sum_{i,k} \Big[\cancel{K_{x_i x_k} L_{y_k} H_{y_i}} - K_{y_k} L_{x_i x_k} H_{y_i} + \cancel{K_{y_i y_k} L_{x_k} H_{x_i}}$$

$$- \cancel{K_{x_k} L_{y_i y_k}} H_{x_i} - K_{y_k x_i} L_{x_k} H_{y_i} + K_{x_k} L_{y_k x_i} H_{y_i}$$

$$- K_{x_k y_i} \cancel{L_{y_k}} H_{x_i} + \cancel{K_{y_k} L_{x_k y_i} H_{x_i}} \Big].$$

and again by symmetry.

$$\{\{L,H\},K\} = \sum_{i,k} \Big[L_{x_i x_k} H_{y_k} K_{y_i} - L_{y_k} H_{x_i x_k} K_{y_i} + \cancel{L_{y_i y_k} H_{x_k} K_{x_i}}$$

$$- \cancel{L_{x_k} H_{y_i y_k}} K_{x_i} - \cancel{L_{y_k x_i} H_{x_k} K_{y_i}} + \boxed{L_{x_k} H_{y_k x_i} K_{y_i}}$$

$$- \cancel{L_{x_k y_i} H_{y_k} K_{x_i}} + L_{y_k} \cancel{H_{x_k y_i}} K_{x_i} \Big].$$

Adding, we find <u>Jacobi's identity</u>

$$(31.1) \qquad \{\{H,K\},L\} + \{\{K,L\},H\} + \{\{L,H\},K\} = 0$$

This reflects the general fact for symplectic manifolds, viz.,

$$(31.2) \qquad d\omega(v_H, v_K, v_L) = c\left(\{\{H,K\},L\} + \{\{K,L\},H\} + \{\{L,H\},K\}\right)$$

for some constant c.

Thus

$$H, K, L \text{ satisfy Jacobi's identity}$$
$$\iff$$
$$\omega \text{ is closed} \quad \text{ie} \quad d\omega = 0$$

Jacobi's identity leads to the following critical calculation for the commutator $[v_H, v_K]$ of Hamiltonian vector fields:

$$
\begin{aligned}
[v_K, v_H](L) &= v_H(v_K(L)) - v_K(v_H(L)) \\
&= v_H(\{L, K\}) - v_K(\{L, H\}) \\
&= \{\{L, K\}, H\} - \{\{L, H\}, K\}. \\
&= \{\{L, K\}, H\} + \{\{H, L\}, K\} \\
&= -\{\{K, H\}, L\} \quad \text{, by Jacobi} \\
&= \{L, \{K, H\}\} \\
&= v_{\{K, H\}}(L)
\end{aligned}
$$

ie

(32.1) $$[v_H, v_K] = v_{\{K, H\}}$$

Thus the commutator of two Hamiltonian vector fields is again a Hamiltonian vector field. Moreover the map

$$H \mapsto v_H$$

is an anti-isomorphism from the Poisson algebra of functions with product given by the Poisson bracket, into the algebra of vector fields with product given by the commutator of vector fields. <u>Most</u> <u>importantly</u>, we notice that two Hamiltonian vector fields commute $\Longleftrightarrow$ their Hamiltonians Poisson commute. This is perhaps the most useful consequence of the fact that ω is closed.

It is a general fact for vector fields on a manifold that

two vector fields $v, \tilde{v}$ commute $\Longleftrightarrow$ the flows $\phi^t_v, \phi^t_{\tilde{v}}$ that they generate commute, i.e. if $\frac{d\phi^t_v}{dt} = v(\phi^t_v)$ and $\frac{d\phi^t_{\tilde{v}}}{dt} = \tilde{v}(\phi^t_{\tilde{v}})$ then

(33.1) $$\phi^t_v \circ \phi^s_{\tilde{v}} = \phi^s_{\tilde{v}} \circ \phi^t_v \qquad \forall s,t .$$

In particular, flows generated by Hamiltonians if and only if their Hamiltonians Poisson commute.

To summarize, we see that symplectic manifolds give rise to (non-degenerate) Poisson manifolds $(M, \{\cdot,\cdot\})$ where $\{\cdot,\cdot\}$

- is bilinear from $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$
- satisfies Leibnitz's rule $\{HK, L\} = H\{K, L\} + \{H, L\}K$
- is non-degenerate ie $\{H, K\} = 0$ for all functions K implies $H = \text{const}$.
- satisfies the Jacobi identity

It is an interesting exercise to prove the converse ie. if $(M, \{\cdot,\cdot\})$ is a non-degenerate Poisson manifold with the four above properties, then (M, ω) is a symplectic manifold where

$$\omega(v_H, v_K) \equiv \{H, K\}$$

and $\quad v_H(L) \equiv \{L, H\}, \quad v_K(L) \equiv \{L, K\},$

<u>Question</u> How do symplectic manifolds arise?

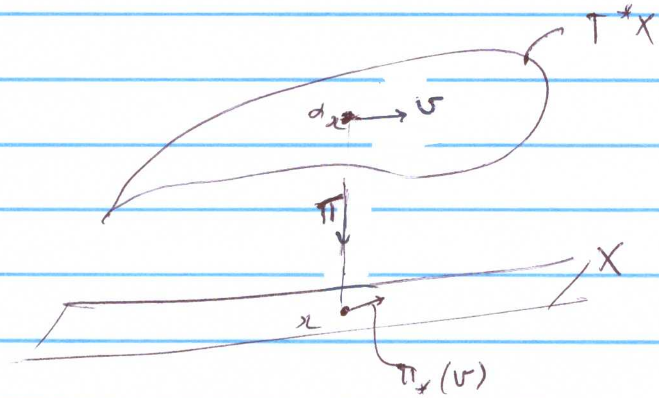
There are three natural sources for symplectic manifolds

(i) T^*X, i.e. cotangent bundles of manifolds

(ii) co-adjoint orbits of groups on their on their dual Lie algebras (here the 2-form is called the Kostant-Kirillov 2-form)

(iii) constrained systems.

(i) Let X be a manifold, let $x \in X$, and let $\alpha_x \in T^*_x X$. Let v be a vector field on T^*X and let π denote the natural projection on T^*X to the base point x: Thus $\pi(\alpha_x) = x$ and $\pi_*(v)$, the push-forward of v under π, lies in $T_x X$



Hence, as $\alpha_x \in T^*_x X$

$$\theta(\alpha_x)(v) \equiv -\alpha_x(\pi_* v)$$

defines a natural 1-form on T^*X. Then $\omega \equiv d\theta$ defines a 2-form on T^*X, which is clearly closed, $d\omega = d^2\theta = 0$, and which can easily be shown to be non-degenerate.

<u>Exercise</u>: On $\mathbb{R}^{2n} = T^*\mathbb{R}^n$, show that

$$\theta = -\sum_{i=1}^{n} y_i\, dx_i$$

and hence

$$\omega = d\theta = \sum_{i=1}^{n} dx_i \wedge dy_i$$

The above construction is very basic in mathematics and leads to the appearance of symplectic manifolds in many different mathematical situations.

vi) Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[\cdot,\cdot]$. Thus $[\cdot,\cdot]$ is

- bilinear, and
- satisfies the Jacobi identity i.e.

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0 \quad \text{for all } x,y,z \in \mathfrak{g}.$$

Let G be the associated connected group and let $\mathfrak{g}^*$ be the dual Lie algebra. Recall that $\mathfrak{g} \cong T_e G$, e = identity element of G. Then G acts on $\mathfrak{g}$ by the

Ad-action

$$Ad : \mathfrak{g} \to \mathfrak{g}$$
$$Ad_g(x) = \frac{d}{dt}\Big|_{t=0} g e^{tx} g^{-1}, \quad g \in G, \ x \in \mathfrak{g}$$

Here $t \mapsto e^{tx}$, $e^{tx}|_{t=0} = e$, is the unique 1-parameter subgroup of G whose tangent vector at e is x

Also G acts on $\mathfrak{g}^*$ by adjointness:

$$Ad^* : \mathfrak{g}^* \to \mathfrak{g}^*$$
$$\langle Ad^*_g \alpha, x\rangle = \langle \alpha, Ad_g x\rangle = \alpha(Ad_g x)$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing of $\mathfrak{g}^*$ with $\mathfrak{g}$.

The co-adjoint orbit O_α through a point $\alpha \in \mathfrak{g}^*$ is given by

(37.1) $$O_\alpha = \{Ad^*_g \alpha : g \in G\}$$

The remarkable fact is that O_α carries a non-degenerate closed 2-form, and hence is naturally a symplectic manifold (and hence is always an even dimensional manifold). To

To see what this 2-form is one can proceed "functorially". Vector fields on $\mathcal{O}_\alpha$ through the point $\beta = Ad^*_g \alpha \in \mathcal{O}_\alpha$ are given by $\frac{d}{dt}\Big|_{t=0} Ad^*_{e^{tx}} \beta$ for arbitrary $x \in \mathfrak{g}$. Indeed

$$\frac{d}{dt}\Big|_{t=0} Ad^*_{e^{tx}} \beta = \frac{d}{dt}\Big|_{t=0} Ad^*_{e^{tx}} Ad^*_g \alpha = \frac{d}{dt}\Big|_{t=0} Ad^*_{ge^{tx}} \alpha \in T_\alpha \mathcal{O}_\alpha$$

Question: How should we define?

$$\omega_\beta \left(\frac{d}{dt}\Big|_{t=0} Ad^*_{e^{tx}} \beta , \frac{d}{dt}\Big|_{t=0} Ad^*_{e^{ty}} \beta \right)$$

The only natural way is

$$\equiv \beta([x,y]) = \langle \beta, [x,y] \rangle$$

and indeed with this definition, $\mathcal{O}_\alpha$ is indeed a symplectic manifold.

Example: $G = Gl^+(n,\mathbb{R}) = Gl(n,\mathbb{R}) \cap \{M : \det M > 0\}$
$\mathfrak{g} = gl(n,\mathbb{R}) = M(n,\mathbb{R})$, the $n \times n$ real matrices.

We can identify $\mathfrak{g}^*$ as $M(n,\mathbb{R})$ through the non-degenerate pairing

(39.1) $$\langle A, B\rangle \equiv \operatorname{tr} AB$$

i.e. the matrix A induces a linear map on $\mathfrak{g}$ to $\mathbb{R}$

through

$$B \mapsto \operatorname{tr}(AB)$$

and moreover, every linear map $\ell(B)$ on $\mathfrak{g}$ to $\mathbb{R}$ is of

this form for some unique $A = A(\ell)$.

Now for $g \in G$, $x \in \mathfrak{g}$, $\operatorname{Ad}_g x = \frac{d}{dt}\Big|_{t=0} g\, e^{tx} g = g x g^{-1}$,

where the RHS can now be taken to be ordinary matrix

multiplication. Also for all $x \in \mathfrak{g} = M(n, \mathbb{R})$

$$\langle \operatorname{Ad}^*_g A, x\rangle = \langle A, \operatorname{Ad}_g x\rangle = \langle A, g x g^{-1}\rangle$$

$$= \operatorname{tr} A g x g^{-1} = \operatorname{tr} g^{-1} A g x.$$

Thus

(39.2) $$\operatorname{Ad}^*_g A = g^{-1} A g \quad \text{and hence}$$

(39.3) $$\mathcal{O}_A = \{g^{-1} A g : g \in G\}$$
$$= \{\text{set of all (real) matrices that are } Gl^+(n, \mathbb{R})\text{-conjugate to } A\}.$$