<u>Lecture 3</u> <u>Exercise</u>: Compute all possible co-adjoint orbits $\mathcal{O}_A$ in the case of 2×2 matrices.

Now observe that each $x \in g$ induces a function H_x on g^* in a natural way, as follows

(40.1) $$H_x(A) = \langle A, x\rangle = \operatorname{tr} Ax \quad , \quad A \in g^*.$$

For such functions H_x we have

$$dH_x\left(\frac{d}{dt}\Big|_{t=0} Ad^*_{e^{ty}} A\right) = \frac{d}{dt}\Big|_{t=0} H_x\left(Ad^*_{e^{ty}} A\right)$$

$$= \frac{d}{dt}\Big|_{t=0} H_x\left(e^{-ty} A e^{ty}\right)$$

$$= \frac{d}{dt}\Big|_{t=0} \operatorname{tr}\left(e^{-ty} A e^{ty} x\right)$$

$$= \operatorname{tr}\left(A[y,x]\right) = \langle A, [y,x]\rangle$$

$$= \omega_A\left(\frac{d}{dt}\Big|_{t=0} Ad^*_{e^{ty}} A \, , \, \frac{d}{dt}\Big|_{t=0} Ad^*_{e^{tx}} A\right)$$

Thus

(40.2) $$V_{H_x}(A) = -\frac{d}{dt}\Big|_{t=0} Ad^*_{e^{tx}} A = -\frac{d}{dt}\Big|_{t=0} e^{-tx} A e^{tx} = [x, A]$$

and

(40.3) $$\{H_x, H_y\} = \omega_A\left(V_{H_x}, V_{H_y}\right) = \operatorname{tr}\left(A[x,y]\right)$$

The differential $dH_x(A)$ is by definition the (unique) functional on $\mathfrak{g}^*$, and hence the unique element in $(\mathfrak{g}^*)^* = \mathfrak{g}$ such that

$$dH_x(A)(B) = \frac{d}{dt}\Big|_{t=0} H_x(A+tB) = \frac{d}{dt}\Big|_{t=0} \operatorname{tr} x(A+tB) = \operatorname{tr} xB$$

Thus

$$dH_x(A) = x$$ and we see that

(41.1) $$\{H_x, H_y\}(A) = \operatorname{tr}\left(A\,[dH_x(A), dH_y(A)]\right)$$

By linearity, approximation (see Exercise below) and Leibnitz rule we conclude that

(41.2) $$\{H, K\}(A) = \operatorname{tr}\left(A\,[dH(A), dK(A)]\right)$$

for arbitrary smooth functions H and K on $\mathfrak{g}^*$

<u>Exercise</u>: Show that arbitrary smooth functions on $\mathfrak{g}^*$ can be approximated by finite linear combinations of sums of powers of functions of type H_x.

On more general dual Lie algebras $\mathfrak{g}^*$ (41.2) becomes

(41.3) $$\{H, K\}(\alpha) = \langle \alpha, [dH(\alpha), dK(\alpha)]\rangle \qquad ; \alpha \in \mathfrak{g}^*$$

where $\langle \alpha, y \rangle$ denotes the action of $\mathfrak{g}^*$ on $\mathfrak{g}$, $\alpha \in \mathfrak{g}^*$ and $y \in \mathfrak{g}$.

In the case of $Gl^+(n, \mathbb{R})$, for $H : M(n, \mathbb{R}) \to \mathbb{R}$, we have

(42.1) $$\frac{d}{dt}\Big|_{t=0} H(A + tB) = \sum_{i,j} \frac{\partial H}{\partial A_{ij}} B_{ij} = \operatorname{tr} \nabla H^T(A) B$$

where $\nabla H(A)$ is the matrix with entries $\left(\frac{\partial H}{\partial A_{ij}}\right)$. Thus

(42.2) $$\{H, K\}(A) = \operatorname{tr}\left(A\,[\nabla H^T(A), \nabla K^T(A)]\right).$$

Notice that what we have really constructed is a <u>Poisson manifold</u> i.e. $\mathfrak{g}^*$ is a manifold with a bracket $\{\cdot,\cdot\}$ which satisfies all the conditions for a Poisson bracket (see p34) <u>except</u> that in general it is degenerate. However $\mathfrak{g}^*$ is <u>foliated</u> by symplectic leaves i.e. sub-manifolds of $\mathfrak{g}^*$ which are are symplectic. These symplectic leaves are precisely the co-adjoint orbits $\mathcal{O}_\alpha$. We will say much more about this in later lectures. In particular we will be interested

Remark: It is important to note that a Hamiltonian K on a Poisson manifold generates a flow via (29.4), $dH/dt = \{H, K\}$, even though the Poisson bracket may be degenerate.

in the group L of lower triangular matrices, its Lie and dual Lie algebras, l & l^*.

Exercise Compute all the symplectic leaves for the dual Lie algebra of $G = Gl^+(2, \mathbb{R})$.

(*) Exercise: Insert from p43+

Exercise In the case that the base manifold X on p35 is a group, say G, the manifolds in (i) and (ii) are related. Basically the symplectic structure in (ii) is the pull back of the structure in (i) to the identity in G. Make this explicit.

(iii) Constrained systems (see, for example, P. Deift, F. Lund and E. Trubowitz, Nonlinear wave equations and constrained harmonic motion, Comm. Math. Phys. 74 (1980), 141-188)

Suppose we have n one-dimensional harmonic oscillators

$$\ddot{x}_i + \lambda_i x_i = 0, \quad 1 \le i \le n.$$

These equations are generated by the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n} (y_i^2 + \lambda_i x_i^2)$$

on the symplectic manifold $(\mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dy_i)$. Suppose we now constrain the oscillators to lie on the sphere

(*) Insert on p43

Exercise

For $G = Gl^+(n, \mathbb{R})$ as above verify directly that

(i) O_α is a manifold immersed in $Gl^+(n, \mathbb{R})$ for any α.

(ii) vectors of the form $(v_x)_\beta = \frac{d}{dt}\Big|_{t=0} Ad^*_{e^{tx}} \beta = \frac{d}{dt}\Big|_{t=0} e^{-tx} \beta e^{tx}$

$= [\beta, x]$, $\beta \in O_\alpha$, span the tangent space to O_α at β

(iii) $\omega_\beta((v_x)_\beta, (v_y)_\beta)$ defined as $\langle \beta, [x, y] \rangle = \mathrm{tr}\, \beta [x, y]$

is indeed a closed, non-degenerate 2 form on O_α, $\beta \in O_\alpha$.

so that O_α is a symplectic manifold.

(iv) Verify directly that $\{H, K\}$ given by (42.2) satisfies the Jacobi identity.

$$\phi_1(x) = \sum_{i=1}^{n} x_i^2 - 1 = 0$$

How would we describe the motion? Well, we would recall how we solved the problem in our first physics course of a particle moving along a wire, and proceed accordingly. The Hamiltonian version of this procedure is the following.

Let

$$\phi_2 = \sum_{i=1}^{n} x_i y_i \quad \left(= \sum_{i=1}^{n} x_i \dot{x}_i \right).$$

Let $X = \{(x,y) \in \mathbb{R}^{2n} : \phi_1 = \phi_2 = 0\}$. Clearly the constrained motion should lie on $X \subset \mathbb{R}^{2n}$. Moreover X is even dimensional and carries a natural 2-form $i^*\omega$, the pull-back of ω under the immersion $i : X \to \mathbb{R}^{2n}$

$$i^*\omega(u,v) = \omega(i_* u, i_* v)$$

where $i_* u$, $i_* v$ are the push-forward of u, v from $T_x X \to T_{i(x)} \mathbb{R}^{2n}$. Recall that if f is a smooth function on

$\mathbb{R}^{2n}$, then $f \circ i$ is a smooth function on X an

$$i_* u(f) = u(f \circ i)$$

Alternatively, $i^* \omega$ is just the restriction of ω to TX $\subset T\mathbb{R}^{2n}$. As the operator d commutes with the restriction (or the pull-back) operation, it is clear that $d(i^*\omega) = 0$ ie $i^*\omega$ is closed. The only question is whether it is non-degenerate.

__Exercise__ Show that $\{\phi_1, \phi_2\} \neq 0$ and conclude that $i_* \omega$ is non-degerate.

__Exercise__ Compute the equations of motion for the above constrained flow. This constrained system is called the Neumann system, and is in fact an "integrable" system (see below)

equation can be obtained by constraining independent Harmonic oscillators to quadrics (see D-L-Trub).

(see Deift-Lund-Trubowitz). It turns out that many well known integrable systems such as the Nonlinear Schrödinger equation, the Sine-Gordon

__Question:__ How can we integrate a dynamical system

(45.1) $$\frac{d}{dt} x(t) = V(x(t)), \quad x(t=0) = x_0,$$

in $\mathbb{R}^m$, say? Suppose we are really lucky and have $m-1$ independent, conserved quantities $\phi_1, \dots, \phi_{m-1}$, so that $\frac{d}{dt}\phi_j(x(t)) = 0$, $j = 1, \dots, m-1$, for solutions $x(t)$ of (45.1). Then we could solve for $m-1$ variables in favor of the first one, say, and then we would be left with the equation $\frac{dx_1}{dt} = V(x_1, x_2(x_1), \dots, x_m(x_1))$ which can be integrated by quadrature

(46.1) $$\int_{(x_0)_1}^{x(t)} \frac{du}{V(u, x_2(u), \dots, x_m(u))} = t$$

Now in the theory of Hamiltonian, as opposed to general, systems a remarkable reduction occurs. One can "solve" systems of dimension $m = 2n$ which have only n independent integrals $\phi_1, \dots, \phi_n$, provided that these integrals have the additional property

(46.2) $$\{\phi_i, \phi_j\} = 0, \quad 1 \le i, j \le n$$

i.e., the integrals Poisson commute. As the Hamiltonian H for the system is conserved, we may always take one of the integrals, say ϕ_1, equal to H. Note that as ϕ_i are integrals of the flow generated by H,

$$\{\phi_i, H\} = \frac{d\phi_i}{dt} = 0$$

so the remaining integrals $\phi_2, \dots, \phi_n$, automatically Poisson commute with $\phi_1 = H$.

A central theorem in the subject is the so-called Liouville-Arnold-Jost Theorem, which says the following:

A Hamiltonian vector field V_H on a symplectic manifold (M^{2n}, ω) is called <u>integrable</u> (or <u>completely integrable</u>) in the sense of L-A-Jost on a domain $D \subset M^{2n}$ if it possesses n integrals $\phi_1 = H$, $\phi_2, \dots, \phi_n$ which are linearly independent on D (i.e. $d\phi_1, \dots, d\phi_n$ are linearly independent at all points of D

and which Poisson commute.

We require that D is invariant under the flow generated by H for all t. Any (Hamiltonian) system is <u>locally</u> integrable i.e. given any H and a point $m \in M^{2n}$, there exists a neighborhood B of m such that H has n Poisson commuting integrals in this neighborhood (<u>Exercise</u>: prove this). But in general the flow generated by H escapes from B, and so these (local) integrals teach us nothing about the global behavior of the flow. The invariance of D is essential.

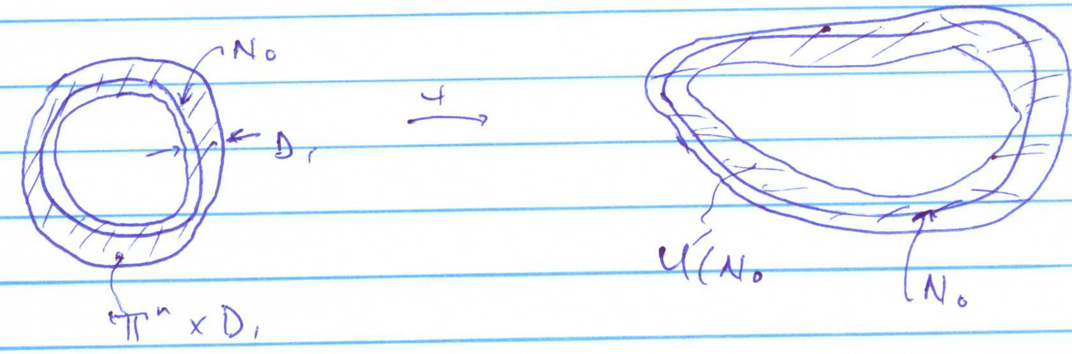
<u>Theorem 48.1</u> (Liouville - Arnold - Jost) (see Moser - Zehnder for a complete proof)

Suppose that $H = \phi_1$ is integrable on a domain $D \subset M^{2n}$ with (Poisson commuting) integrals $\phi = \{\phi_1, \dots, \phi_n\}$ and suppose that $N_0 = \phi^{-1}(0) \subset M^{2n}$ is compact and connected. Then

(a) N_0 is an imbedded n-dimensional torus $\mathbb{T}^n$

(b) There exists an open neighborhood $U(N_0) \subset M^{2n}$ of N_0 which can be coordinatized as follows: if $x = \{x_1, \ldots, x_n\}$ are variables on the torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ and $y = (y_1, \ldots, y_n) \in D_1$, where $D_1 \subset \mathbb{R}^n$ is a domain containing the origin 0, there exists a diffeomorphism

$$\psi : \mathbb{T}^n \times D_1 \to U(N_0)$$



Moreover ψ is symplectic, i.e. $\psi^* \omega = \sum_{i=1}^n dx_i \wedge dy_i$, or equivalently, $\{K, L\}_{(M^{2n}, \omega)} \circ \psi = \{K \circ \psi, L \circ \psi\}_{(\mathbb{T}^n \times D_1, \sum_i dx_i \wedge dy_i)}$, and

(49.1) $$H \circ \psi = h(y_1, \ldots, y_n)$$

for some function h.

In particular, near a compact, connected level set $\phi^{-1}(0)$, the flow generated by H is extremely simple, in in terms of the variables $(x,y) \in \mathbb{T}^n \times D_1$. Now it is a **Exercise** to show that if χ is a symplectic diffeomorphism (i.e. a "sympletomorphism") from one symplectic manifold (M_1, ω_1) onto a second symplectic manifold (M_2, ω_2), then if $t \mapsto m_2(t)$ is a Hamiltonian flow on (M_2, ω_2) generated by a Hamiltonian $H_2 : M_2 \to \mathbb{R}$, then $t \mapsto m_1(t) \equiv \chi^{-1}(m_2(t))$ is a Hamiltonian flow on (M_1, ω_1) generated by the Hamiltonian $H_1 = H_2 \circ \chi$. Thus as ψ is symplectic from $\mathbb{T}^n \times D' \to (M, \omega)$, the flow $m(t)$ generated by H on (M, ω) is transformed to a Hamiltonian flow $(x(t), y(t))$ on $\mathbb{T}^n \times D'$ generated by the Hamiltonian $h = H \circ \psi$. Thus

$$\dot{x}_j = \frac{\partial}{\partial y_j} H \circ \psi = \frac{\partial}{\partial y_j} h$$

$$\dot{y}_j = -\frac{\partial}{\partial x_j} H \circ \psi = -\frac{\partial h}{\partial x_j} = 0$$

so that

(50.1) $$x_j(t) = x_j(0) + t \frac{\partial h}{\partial y_j}(y_1, \dots, y_n), \quad y_j(t) = y_j(0)$$

Thus the system can be integrated explicitly and is given by a straight line motion on a torus. The variables $\{y_i\}$ are called the *actions* for the system and the $\{x_i\}$ are called the angles. Returning to the variables for the flow on (M, ω), we have

(51.2) $$m(t) = \chi\left(x(0) + t\, h_y(y(0)),\ y(0) \right)$$

The Theorem not only tells us how to integrate the system (in terms of the variables (x, y), which may or may not be hard to construct), but perhaps more importantly, we can understand the qualitative behavior of the system, as we now explain.

The quantities $w = (w_1, \dots, w_n) = \left(\frac{\partial h}{\partial y_1}, \dots, \frac{\partial h}{\partial y_n}\right)$ are called the *frequencies* of the system. The neighborhood $U(N_0)$ in the Theorem is foliated by tori which we parameterized by the values of $y \in D$. As $w = w(y)$, and as y is conserved by the flow, the frequencies are constant on the torus. In general, however, they vary from torus to torus. We can distinguish tori according to whether the w_i's are rationally independent or rationally dependent.

Rationally independent: Here $\sum_{i=1}^{n} j_i w_i = 0$ for $j_i \in \mathbb{Z}$ $\Rightarrow$ $j_i = 0$. In this case, by a well-known theorem of Kronecker, $\{x_0 + t w : t \in \mathbb{R}\}$ is dense in $\mathbb{T}^n$. Thus the orbit of the flow is dense on such tori and in fact the flow is quasiperiodic in time with n frequencies.

Rationally dependent: Here $\sum_{i=1}^{n} j_i w_i = 0$ for some $j_i \in \mathbb{Z}$, not all zero. Then the flow is restricted to a sub-torus of $\mathbb{T}^n$. For example, if $n=2$ and $w_1 - 2w_2 = 0$, then the flow is restricted to $\{(x_1, x_2) \in \mathbb{T}^2 : x_1 - 2x_2 = \text{const} \pmod{\mathbb{Z}}\}$.

Again the flow is almost periodic, but with fewer frequencies.

Thus the essential problem of describing the long-time behavior of integrable Hamiltonian systems is solved, in principle, provided the invariant set $\phi^{-1}(0)$ is compact and connected. If $\phi^{-1}(0)$ is compact but not connected, we can just restrict our attention to each connected component. Also if $\phi^{-1}(0)$ is not compact, then the theorem goes through provided it is known a priori that each ϕ_i generates a global flow, at least for data near $\phi^{-1}(0)$. In this case one learns that $\phi^{-1}(0)$ has a neighborhood $U(N_0)$ which is a thickening by an n-dimensional disk D_1 of a product of lines and circles

$$U(N_0) = \mathbb{T}^k \times \mathbb{R}^{n-k} \times D_1$$

On each leaf $\mathbb{T}^k \times \mathbb{R}^{n-k} \times \{y\}$ the flow is again given by a straight line motion, but now the winding takes place on a cylinder rather than a torus.

The Liouville - Arnold - Jost Theorem describes qualitatively the behavior of an integrable system. In each case the serious and essential task remains to determine the action angle variables explicitly. As noted above, full details of the proof of the L - A - Jost Theorem are given in Mos-Zehn, but here we give a <u>Sketch of the proof of The Theorem:</u>

As $\{\phi_i, \phi_j\} = 0$, $[v_{\phi_i}, v_{\phi_j}] = v_{\{\phi_j, \phi_i\}} = 0$, by (32.1);

which shows that the Hamiltonian vector fields v_{ϕ_i} induced by the ϕ_i's commute, and hence the flows $t \mapsto \psi_i^t = \psi_i(t, m)$, $\psi_i(0,m) = m$, induced by the v_{ϕ_i}'s commute (see (33.11)). This means that we can immerse $\mathbb{R}^{2n}$ into M^{2n} as follows. Fix $m_0 \in \Phi^{-1}(0)$. Then the map

$$\mathbb{R}^n \ni t = (t_1, \dots, t_n) \longmapsto T(t) = \psi_1^{t_1} \circ \cdots \circ \psi_n^{t_n}(m)$$

takes $\mathbb{R}^n$ into the level set $N_0 = \{m : \phi_i(m) = \phi_i(m_0),\ i = 1, \dots, n\}$.
This is because $\frac{d}{dt_j}\phi_i(\varphi_j^{t_j}(m)) = \{\phi_i, \phi_j\}(\varphi_j^{t_j}(m)) = 0$, so that ϕ_i is a constant of the motion for all the ϕ_j-flows, $1 \le i, j \le n$.
A simple argument shows that T is onto N_0. Let $\Lambda = \{t \in \mathbb{R}^n : T(t) = m_0\}$. Now as the flows commute it follows that Λ is a lattice in $\mathbb{R}^n$. On the other hand, $\mathbb{R}^n/\Lambda$ is mapped diffeomorphically onto N_0. But N_0 is compact by assumption. Now the lattice Λ has k generators, $1 \le k \le n$, i.e. $\Lambda = \{j_1\hat{v}_1 + \dots + j_k\hat{v}_k : j_i \in \mathbb{Z}\}$, where $\hat{v}_i \in \mathbb{R}^n$. If $k < n$, then clearly $\mathbb{R}^n/\Lambda$ cannot be compact which is a contradiction as $N_0 \cong \mathbb{R}^n/\Lambda$ is compact. Hence we must have $k = n$: But then $\mathbb{R}^n/\Lambda \cong N_0$ is clearly a torus. Then we must "thicken" things around N_0, $N_0 \to U(N_0), \dots$ $\square$

We now consider some elementary examples of integrable systems.

~~$M = \mathbb{R}^2$, $H = \frac{1}{2}(p^2 + \omega^2 q^2) = \phi_1$~~

~~Claim: H is integrable on the domain $D = M/\{0\} = \mathbb{R}^2/\{0\}$~~