Lecture 6

Now as $da_1/dt = 2b_1^2(t)$ $\forall t$, we see that $a_1(t)$ is monotone decreasing as $t \to -\infty$, so that, as above,

$$a_1(-\infty) = \lim_{t\to-\infty} a_1(t)$$

exist and

$$b_1(t) \to 0 \qquad \text{as } t \to -\infty.$$

Continuing we find that

$$a_k(-\infty) = \lim_{t\to\infty} a_k(t) \qquad , k = 1, \dots, N$$

exist and

$$b_k(t) \to 0 \qquad \text{as } t \to -\infty \qquad , k = 1, \dots, N-1$$

Necessarily the $a_k(-\infty)$'s are the eigenvalues of $X(0)$

Arguing as above, we have for $k = 1, \dots, N$

(95.1) $$x_k(t) = -2a_k(-\infty)t + o(t) \qquad \text{as } t \to$$

and so for $k = 1, \dots, N-1$

(95.2) $$x_k(t) - x_{k+1}(t) = -2(a_k(-\infty) - a_{k+1}(-\infty))t + o(t)$$

But again as $b_k = \frac{1}{2} e^{\frac{1}{2}(x_k(t) - x_{k+1}(t))} \to 0$ as $t \to -\infty$, we must have

$$a_k(-\infty) - a_{k+1}(-\infty) < 0$$

It follows that we must hat

(96.1) $$a_k(-\infty) = \lambda_{N-k+1}, \qquad 1 \le k \le N$$

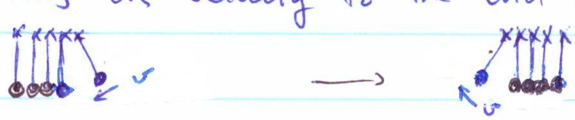
Thus we have a billiard ball type interaction: As $t \to -\infty$,

$$\underset{x_1}{\overset{\xleftarrow{-2\lambda_N t}}{\times}} \quad \underset{x_2}{\overset{\xleftarrow{-2\lambda_{N-1} t}}{\times}} \quad \cdots \quad \underset{x_{N-1}}{\overset{\xleftarrow{-2\lambda_2 t}}{\times}} \quad \underset{x_N}{\overset{-2\lambda_1 t}{\times}}$$

and as $t \to \infty$

$$\underset{x_1}{\overset{\xleftarrow{-2\lambda_1 t}}{\times}} \quad \underset{x_2}{\overset{\xleftarrow{-2\lambda_2 t}}{\times}} \quad \cdots \quad \underset{x_{N-1}}{\overset{-2\lambda_{N-1} t}{\times}} \quad \underset{x_N}{\overset{-2\lambda_N t}{\times}}.$$

So the particle x_N, which as velocity $-2\lambda_1$ as $t \to -\infty$, transfers this velocity to x_1 as $t \to +\infty$, etc. This is reminiscent of a ball impinging on a row of balls with velocity v, and transfering its velocity to the end ball after collision



before impact $\longrightarrow$ after impact

Finally observe that from (93.1) and (95.1), $b_k(t) \to 0$ exponential fast.

We now compute the error term $o(t)$ in (95.1) more precisely. This requires a more detailed approach.

From the theory of skew-symmetric tensors we have for vectors $u_0, u_1, \ldots, u_k$ in $\mathbb{R}^N$

(97.1)
$$(u_0 \wedge u_1 \wedge \cdots \wedge u_k,\ u_0 \wedge u_1 \wedge \cdots \wedge u_k)$$
$$= \det\big((u_i, u_j)\big)_{0 \le i, j \le k}.$$

In particular for

(97.2)
$$u_i = X_0^{\,i} e_1 \quad , \quad 0 \le i \le k$$

where $X_0 = X(0)$ and $e_1 = (1, 0, \ldots, 0)^T$

$$(e_1 \wedge X_0 e_1 \wedge \cdots \wedge X_0^k e_1,\ e_1 \wedge X_0 e_1 \wedge \cdots \wedge X_0^k e_1)$$

$$= \det \begin{pmatrix} (e_1, e_1) & (e_1, X_0 e_1) \cdots & (e_1, X_0^k e_1) \\ (X_0 e_1, e_1) & (X_0 e_1, X_0 e_1) & (X_0 e_1, X_0^k e_1) \\ \vdots & & \\ (X_0^k e_1, e_1) & & (X_0^k e_1, X_0^k e_1) \end{pmatrix}$$

$$= \det \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & & \\ \vdots & & & \\ c_n & & & c_{2n} \end{pmatrix} \equiv D_n .$$

Where

(98.1) $$c_j = (X_0^j e_1, e_1) = \sum_{i=1}^{N} \lambda_i^j \, u_i^2(1)$$

where $\{\lambda_i\}$ are the eigenvalues of X_0 and $u_i(1) > 0$ are the first components of the corresponding eigenvectors.

Rewriting, we have

(98.2) $$c_j = \int \lambda^j \, d\mu(\lambda)$$

where

(98.3) $$d\mu(\lambda) = \sum_{i=1}^{N} u_i^2(1) \, \delta_{\lambda_i}(\lambda)$$

On the other hand as $X_0 = \begin{pmatrix} a_1 & b_1 & & 0 \\ b_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{N-1} \\ 0 & & b_{N-1} & a_N \end{pmatrix}$ is a Jacobi matrix,

$$X_0 e_1 = b_1 e_2 + a_1 e_1 \quad = b_1 e_1 + r_1 , \qquad r_1 \in \langle e_1 \rangle$$

$$X_0^2 e_1 = b_1 X_0 e_2 + a_1 X_0 e_1$$
$$= b_1 b_2 e_3 + r_2 , \qquad r_2 \in \langle e_1, e_2 \rangle$$

and by induction

(98.4) $$X_0^k = b_1 b_2 \cdots b_k e_{k+1} + r_k , \quad \text{where } r_k \in \langle e_1, \dots, e_k \rangle .$$

Thus

$$e_1 \wedge X_0 e_1 \wedge \cdots \wedge X_0^k e_1$$

$$= e_1 \wedge (b_1 e_2 + r_1) \wedge (b_1 b_2 e_3 + r_2) \wedge \cdots \wedge (b_1 \cdots b_k e_{k+1} + r_k)$$

$$= b_1 \left(e_1 \wedge e_2 \wedge (b_1 b_2 e_3 + r_2) \wedge \cdots \wedge (b_1 \cdots b_k) e_{k+1} + r_k \right)$$

$$= b_1 (b_1 b_2) \left(e_1 \wedge e_2 \wedge e_3 \wedge \cdots \wedge (b_1 \cdots b_k) e_{k+1} + r_k \right)$$

$$= b_1 (b_1 b_2)(b_1 b_2 b_3) \cdots (b_1 \cdots b_k)(e_1 \wedge e_2 \wedge \cdots \wedge e_k)$$

Thus

(99.1) $$D_k = b_1^2 (b_1 b_2)^2 \cdots (b_1 \cdots b_k)^2, \quad 1 \le k \le N-1,$$

and

(99.2) $$D_0 = 1$$

Hence for $3 \le k \le N-1$

(99.3) $$\frac{D_k D_{k-2}}{D_{k-1}^2} = \frac{b_1^2 (b_1 b_2)^2 \cdots (b_1 \cdots b_k)^2 \quad b_1^2 (b_1 b_2)^2 \cdots (b_1 \cdots b_{k-2})^2}{b_1^4 (b_1 b_2)^4 \cdots (b_1 \cdots b_{k-1})^4}$$

$$= \frac{(b_1 \cdots b_{k-1})^2 (b_1 \cdots b_k)^2}{(b_1 \cdots b_{k-1})^4}$$

$$= b_k^2$$

Also for $k=2$

$$\frac{D_2 D_0}{D_1^2} = \frac{b_1^2 (b_1 b_2)^2 \cdot 1}{b_1^4} = b_2^2$$

and for $k=1$

$$\frac{D_1 D_{-1}}{D_0^2} = b_1^2$$

where $\theta_{-1} \equiv 1$

Thus

(100.1) $$b_k^2 = \frac{\theta_k \theta_{k-2}}{\theta_{k-1}^2}, \quad 1 \le k \le N-1,$$

provided $\theta_{-1} \equiv 1$

Note from (98.1) that (100.1) expresses b_k, and hence $x_k - x_{k-1}$, directly in terms of the data $(\lambda_1, \dots, \lambda_N)$ and $(u_1(1), \dots, u_N(1))$, whose evolution under the isospectrality and Toda flow is known explicitly from (86.3).

Now we expand θ_k into a more useful formula as follows: By (98.3) and the formula for a vander Monde determinant,

$$\theta_k = \det \begin{pmatrix} \int d\mu(x_0) & \int x_1 \, d\mu(x_1) & \cdots & \int x_k^k \, d\mu(x_k) \\ \int x_0 \, d\mu(x_0) & \int x_1^2 \, d\mu(x_1) & & \int x_k^{k+1} \, d\mu(x_k) \\ \vdots & \vdots & & \vdots \\ \int x_0^k \, d\mu(x_0) & \int x_1^{k+1} \, d\mu(x_1) & & \int x_k^{2k} \, d\mu(x_k) \end{pmatrix}$$

$$= \int \cdots \int d\mu(x_0) \cdots d\mu(x_k) \; \det \begin{pmatrix} 1 & x_1 & \cdots & x_k^k \\ x_0 & x_1^k & & x_k^{k+1} \\ \vdots & \vdots & & \vdots \\ x_0^k & x_1^{k+1} & & x_k^{2k} \end{pmatrix}$$

$$= \int \cdots \int q\mu(x_0) \cdots q\mu(x_k)\, x_0^0 x_1^1 \cdots x_k^k \det\begin{pmatrix} 1 & \cdots & 1 \\ x_0 & & x_k \\ \vdots & & \vdots \\ x_0^k & & x_k^k \end{pmatrix}$$

$$= \int \cdots \int q\mu(x_0) \cdots q\mu(x_k)\, x_0^0 x_1^1 \cdots x_k^k \prod_{0 \le \ell < j \le k} (x_j - x_\ell)$$

Letting the permutation group act on $0, 1, \ldots, k$, we find

$$D_k = \frac{1}{(k+1)!} \sum_{\substack{\text{permutations} \\ \pi}} \int \cdots \int q\mu(x_{\pi(0)}) \cdots q\mu(x_{\pi(k)})\, x_{\pi(0)}^0 \cdots x_{\pi(k)}^k$$

$$\times \left(\text{action of } \pi \text{ on the vanderMonde determinant}\right)$$

But the action of π on the vanderMonde determinant is given by

$$\det\begin{pmatrix} x_{\pi(0)}^0 & x_{\pi(1)}^0 & \cdots & x_{\pi(k)}^0 \\ \vdots & \vdots & & \vdots \\ x_{\pi(0)}^k & x_{\pi(1)}^k & & x_{\pi(k)}^k \end{pmatrix}$$

$$= \operatorname{sgn}\pi \det\begin{pmatrix} x_0^0 & \cdots & x_k^0 \\ x_0^1 & & x_k^1 \\ \vdots & & \vdots \\ x_0^k & & x_k^k \end{pmatrix}$$

Thus

$$D_k = \frac{1}{(k+1)!} \sum_{\pi} \int q\mu(x_{\pi(0)}) \cdots q\mu(x_{\pi(k)})\, x_{\pi(0)}^0 x_{\pi(1)}^1 \cdots x_{\pi(k)}^k \operatorname{sgn}\pi$$

$$\times \prod_{k \ge j > \ell \ge 0} (x_j - x_\ell)$$

$$= \frac{1}{(h+1)!} \int \cdots \int d\mu(x_0) \cdots d\mu(x_h) \left(\sum_{\pi} \operatorname{sgn} \pi \; x^0_{\pi(0)} \cdots x^h_{\pi h} \right) \prod_{0 \le l < j \le h} (x_j - x_l)$$

ie for $h \geq 1$

(102.1) $$D_h = \frac{1}{(h+1)!} \int \cdots \int d\mu(x_0) \cdots d\mu(x_h) \prod_{0 \le l < j \le h} (x_j - x_l)^2$$

$$= \int \cdots \int_{x_0 > x_1 > \cdots > x_h} d\mu(x_0) \cdots d\mu(x_h) \prod_{0 \le l < j \le h} (x_j - x_l)^2 .$$

as $\prod_{0 \le l < j \le N} (x_j - x_l)^2$ is invariant under permutations and terms with $x_i = x_{i+1}$ for some i, just drop out. As before $D_0 = 1$.

Now under the Toda flow

(102.3) $$d\mu(\lambda, t) = \sum_{i=1}^{N} u_i^2(1,t)\, \delta_{\lambda_i}(\lambda)$$

$$= \frac{\sum_{i=1}^{N} u_i^2(1,0) e^{2\lambda_i t} \delta_{\lambda_i}(\lambda)}{\sum_{i=1}^{N} u_i^2(1,0) e^{2\lambda_i t}}$$

$$= \frac{e^{2\lambda t} \, d\mu(\lambda)}{\int e^{2s t} d\mu(s)}$$

where $d\mu(\lambda)$ is given by (98.3)

Thus under the Toda flow, for $h \geq 1$

$$(103.1)\qquad D_k(t) = \frac{1}{\left(\int e^{2xt}\, d\mu(x)\right)^{k+1}} \int\cdots\int_{x_0>\cdots>x_k} d\mu(x_0)\cdots d\mu(x_k)\, e^{2(x_0+\cdots+x_k)t} \prod_{0\le l<j\le k} (x_j - x_l)^2$$

and so from (100.1), for $k>2$,

$$(103.2)\qquad b_k^2 = \left(\int e^{2xt}\, d\mu(x)\right)^{-(k+1)-(k-1)+2k}$$

$$\times \left(\int\cdots\int_{x_0>\cdots>x_k} d\mu(x_0)\cdots d\mu(x_k)\, e^{2(x_0+\cdots+x_k)t} \prod_{0\le l<j\le k} (x_j - x_l)^2\right)$$

$$\times \int\cdots\int_{x_0>\cdots>x_{k-2}} d\mu(x_0)\cdots d\mu(x_{k-2})\, e^{2(x_0+\cdots+x_{k-2})t} \prod_{0\le l<j\le k-2} (x_j - x_l)^2$$

$$\times \left(\int\cdots\int_{x_0>\cdots>x_{k-1}} d\mu(x_0)\cdots d\mu(x_{k-1})\, e^{2(x_0+\cdots+x_{k-1})t} \prod_{0\le l<j\le k-1} (x_j - x_l)^2\right)^{-2}$$

Thus for $k>2$

$$\cancel{(103.3)\qquad b_k^2 = \frac{\left(\int\cdots\int_{x_0>\cdots>x_k} d\mu(x_0)\cdots d\mu(x_k)\, e^{2(x_0+\cdots+x_k)t}\, V_k^2(x)\right)\left(\int\cdots\int_{x_0>\cdots>x_{k-2}} d\mu(x_0)\cdots d\mu(x_{k-2})\, e^{2(x_0+\cdots+x_{k-2})t}\, V_{k-2}^2(x)\right)}{\int\cdots\int_{x_0>\cdots>x_{k-1}} d\mu(x_0)\cdots d\mu(x_k}}$$

$$(103.3)\qquad b_k^2 = \frac{\left(\int\cdots\int_{x_0>\cdots>x_k} d\mu^k(x)\, e^{2(x_0+\cdots+x_k)t}\, V_k^2(x)\right)\left(\int\cdots\int_{x_0>\cdots>x_{k-2}} d\mu^{k-2}(x)\, e^{2(x_0+\cdots+x_{k-2})t}\, V_{k-2}^2(x)\right)}{\int\cdots\int_{x_0>\cdots>x_{k-1}} d\mu^{k-1}(x)\, e^{2(x_0+\cdots+x_{k-1})t}\, V_{k-1}^2(x)}$$

$k>2$.

where

(104.1) $\quad qu^k(x) = qu(x_0)\cdots qu(x_k)$ and $V_k(x) = $ van der Monde $= \prod_{0\le l<j\le k} (x_j - x_l)$, $k\ge 1$

$= 1 \qquad k = 0.$

For $k=2$, from (100.1)

(104.2)
$$b_2^2 = \frac{D_2 D_0}{(D_1)^2} = \frac{D_2}{(D_1)^2} = \left(\int e^{2xt} qu(x)\right)^{-3}\left(\int e^{2xt} qu(x)\right)^4 \frac{\displaystyle\int_{x_0>x_1>x_2} qu(x_0)qu(x_1)qu(x_2)\, e^{2(x_0+x_1+x_2)t}\, V_2^2(x)}{\left(\displaystyle\iint_{x_0>x_1} qu(x_0)qu(x_1)\, e^{2(x_0+x_1)t}\, V_1^2(x)\right)^2}$$

and for $k=1$

(104.1)
$$b_1^2 = \frac{D_1 \times 1}{1^2} = \frac{1}{\left(\int e^{2xt} qu(x)\right)^2}\iint_{x_0>x_1} qu(x_0)qu(x_1)\, e^{2(x_0+x_1)t}\, V_1^2(x)$$

Expanding we obtain

$$b_1^2 = \frac{1}{2}\iint qu(x_0,t)\, qu(x_1,t)\,(x_0-x_1)^2$$

$$= \frac{1}{2}\left[2\int qu(x_0,t)\, x_0^2 - 2\left(\int qu(x,t)\, x\right)^2\right]$$

$$= \int x^2 qu(x,t) - \left(\int x\, qu(x,t)\right)^2 \qquad \text{---}(*)$$

Recall from (71.2) $b_1^2(t) = \sum_{i=1}^N (\lambda_i - a_1)^2 u_i^2(1,t) = \sum_i^N \lambda_i^2 u_i^2(1,t) - \left(\sum_i^N \lambda_i u_i^2(1,t)\right)^2$

which matches $(*)$, but (104.1) yields the leading behavior of

$b_1(t)$ directly. Indeed as $t \to \infty$, noting that $V_k(x) = 0$ if $x_i = x_j$ for some $i \neq j$,

(105.4) $$b_1^2(t) = \frac{1}{\left(e^{2\lambda_1 t} u_1^2(1) + O(e^{2\lambda_2 t})\right)^2}$$

$$\times \quad u_1^2(1)\, u_2^2(1)\, e^{2(\lambda_1 + \lambda_2)t} (\lambda_1 - \lambda_2)^2 + O\left(e^{(\lambda_1 + \lambda_3)t}\right)$$

$$= \frac{u_2^2(1)}{u_1^2(1)} e^{-2(\lambda_1 - \lambda_2)t} (\lambda_1 - \lambda_2)^2 \left(1 + \text{exp. small}\right).$$

which agrees with (89.3).

Now for $k > 2$, we have from (105.3), as $t \to \infty$

$$b_k^2(t) = \frac{\left(u_1^2(1) \cdots u_{k+1}^2\, e^{2(\lambda_1 + \cdots + \lambda_{k+1})t} \prod_{k \geq j > \ell \geq 0} (\lambda_{j+1} - \lambda_{\ell+1})^2 \right) \times \left(u_1^2(1) \cdots u_{k-1}^2(1)\, e^{2(\lambda_1 + \cdots + \lambda_{k-1})t} \prod_{k-2 \geq j > \ell \geq 0} (\lambda_{j+1} - \lambda_{\ell+1})^2 \right)}{\left(u_1^2(1) \cdots u_k^2(1)\, e^{2(\lambda_1 + \cdots + \lambda_k)t} \prod_{k-1 \geq j > \ell \geq 0} (\lambda_{j+1} - \lambda_{\ell+1})^2 \right)^2}$$

Thus for $3 \leq k \leq N-1$

(105.1) $$b_k^2(t) = \frac{u_{k+1}^2(1)}{u_k^2(1)} e^{2(\lambda_{k+1} - \lambda_k)t} \frac{\prod_{\ell=0}^{k-1} (\lambda_{k+1} - \lambda_{\ell+1})^2}{\prod_{\ell=0}^{k-2} (\lambda_k - \lambda_{\ell+1})^2} \left(1 + \text{exp. small}\right)$$

For $k=2$, from (104.2) we have, as $t\to\infty$,

$$b_2^2(t) = \left(e^{2\lambda_1 t} u_1^2(1) + \text{exp-small}\right)$$

$$\times \frac{u_1(1)^2 u_2^2(1) u_3^2(1)\, e^{2(\lambda_1+\lambda_2+\lambda_3)t}\left((\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)\right)^2}{\left(u_1(1)^2 u_2(1)^2\, e^{2(\lambda_1+\lambda_2)t}(\lambda_1-\lambda_2)^2\right)^2}$$

$$\times (1+\text{exp. small}).$$

$$= \frac{u_3^2(1)}{u_2(1)^2}\, e^{2(\lambda_3-\lambda_2)t}\, \frac{\prod_{\ell=0}^{1}(\lambda_3-\lambda_{\ell+1})^2}{\sum_{\ell=0}^{0}(\lambda_2-\lambda_{\ell+1})^2}\,(1+\text{exp. small})$$

Thus we see that (105.1) also holds for $k\geq 2$, and by (105.3) also holds for $k=1$, provided we interpret

(106.0) $$\prod_{\ell=0}^{-1}(\lambda_1-\lambda_{\ell+1})^2 \equiv 1.$$

<u>Exercise</u> Compute the analogous formulae for $a_k(t)$, $t\to\infty$.

A similar calculation shows that as $t\to-\infty$,

(106.1) $$b_k^2(t) = \frac{u_{N-k}^2(1)}{u_{N-k+1}^2(1)}\, \frac{\prod_{\ell=N-k+1}^{N}(\lambda_{N-k}-\lambda_\ell)^2}{\prod_{\ell=N-k+2}^{N}(\lambda_{N-k+1}-\lambda_\ell)^2}\, e^{2t(\lambda_{N-k}-\lambda_{N-k+1})}$$

$$\times (1+\text{exp. small})$$

Again, this formula holds for $k = 1, \ldots, N-1$, provided

we interpret the denominator in (106.1) as 1, i.e.

$$b_1^2(t) = \frac{u_{N-1}^2(1)}{u_N^2(1)} (\lambda_{N-1} - \lambda_N)^2 e^{2t(\lambda_{N-1} - \lambda_N)} (1 + \text{exp. small}).$$

We now convert the asymptotic formulae for the '$b_k(t)$'s

into asymptotic formulae for $x_k(t)$ as $t \to \pm\infty$. We have as $t \to +\infty$,

(107.1) $$x_k - x_{k+1} = 2\log 2 + 2\log b_k$$

$$= 2t(\lambda_{k+1} - \lambda_k) + 2\log\left(2\,\frac{u_{k+1}(1)}{u_k(1)}\, \frac{\prod_{l=1}^{k} (\lambda_l - \lambda_{k+1})}{\prod_{l=1}^{k-1} (\lambda_l - \lambda_k)} \right) + o(1)$$

and as $t \to -\infty$

(107.2) $$x_k - x_{k+1} = 2t(\lambda_{N-k} - \lambda_{N-k+1}) + 2\log\left(2\,\frac{u_{N-k}(1)}{u_{N-k+1}(1)}\, \frac{\prod_{l=N-k+1}^{N} (\lambda_{N-k} - \lambda_l)}{\prod_{l=N-k+2}^{N} (\lambda_{N-k+1} - \lambda_l)} \right) + o(1)$$

Now
$$\frac{d}{dt}\sum_{k=1}^{N} x_k = \sum_{k=1}^{N} y_k = -2\sum_{k=1}^{N} a_k = -2\sum_{i=1}^{N} \lambda_i = \text{const}$$
and so

(107.3) $$\sum_{k=1}^{N} x_k(t) = \sum_{k=1}^{N} x_k(0) - 2\left(\sum_{i=1}^{N} \lambda_i\right) t$$

Summing (107.1), we obtain as $t \to +\infty$ for $1 \le k \le N-1$, (108.1) $$x_k - x_N = 2t(\lambda_N - \lambda_k) + (\log 4)(N-k)$$

$$+ 2\log\left(\frac{u_N(1)}{u_k(1)} \frac{\prod_{\ell=1}^{N-1}(\lambda_\ell - \lambda_N)}{\prod_{\ell=1}^{k-1}(\lambda_\ell - \lambda_k)}\right) + O(1)$$

and so adding over k

$$\sum_{k=1}^{N-1} x_k - (N-1)x_N = 2t\left((N-1)\lambda_N - \sum_{k=1}^{N-1}\lambda_k\right)$$

$$+ \log 4 \sum_{k=1}^{N-1}(N-k)$$

$$+ 2\sum_{k=1}^{N-1} \log\left(\frac{u_N(1)}{u_k(1)} \frac{\prod_{\ell=1}^{N-1}(\lambda_\ell - \lambda_N)}{\prod_{\ell=1}^{k-1}(\lambda_\ell - \lambda_k)}\right) + O(1)$$

Inserting (107.3), we find that as $t \to +\infty$

$$x_N(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(0) - 2t\lambda_N - \frac{\log 4}{N}\left(\sum_{k=1}^{N-1} N-k\right)$$

$$- \frac{2}{N}\sum_{k=1}^{N-1} \log \frac{u_N(1)}{u_k(1)} \frac{\prod_{\ell=1}^{N-1}(\lambda_\ell - \lambda_N)}{\prod_{\ell=1}^{k-1}(\lambda_\ell - \lambda_k)} + O(1)$$

which implies using (108.1) that as $t \to +\infty$

(108.2) $$x_k(t) = -2t\lambda_k + \frac{1}{N}\sum_{k=1}^{N} x_k(0) - \frac{\log 4}{N}\sum_{k=1}^{N-1}(N-k) + (\log 4)(N-k)$$

$$- \frac{2}{N}\sum_{k=1}^{N-1} \log\left(\frac{u_N(1)}{u_k(1)} \frac{\prod_{\ell=1}^{N-1}(\lambda_\ell - \lambda_N)}{\prod_{\ell=1}^{k-1}(\lambda_\ell - \lambda_k)}\right) + 2\log \frac{u_N(1)}{u_k(1)} \frac{\prod_{\ell=1}^{N-1}(\lambda_\ell - \lambda_N)}{\prod_{\ell=1}^{k-1}(\lambda_\ell - \lambda_k)}$$

$$+ O(1)$$

which reduces after some elementary algebra to

(109.1) $$x_k(t) = -2t\lambda_k + \frac{1}{N}\sum_{j=1}^{N} x_j(0) - \frac{2}{N}\sum_{j=1}^{N}\log\left(\frac{u_k(1)}{u_j(1)}\,\frac{\prod_{l=1}^{k-1}(2\lambda_l - 2\lambda_k)}{\prod_{l=1}^{j-1}(2\lambda_l - 2\lambda_j)}\right) + o(1)$$

as $t \to +\infty$, and a similar calculation using

(107.2) yields

(109.2) $$x_k(t) = -2t\lambda_{N-k+1} + \frac{1}{N}\sum_{j=1}^{N} x_j(0) - \frac{2}{N}\sum_{j=1}^{N}\log\left(\frac{u_{N-k+1}(1)}{u_{N-j+1}(1)}\,\frac{\prod_{l=N-k+2}^{N}(2\lambda_{N-k+1} - 2\lambda_l)}{\prod_{l=N-j+2}^{N}(2\lambda_{N-j+1} - 2\lambda_l)}\right) + o(1)$$

as $t \to -\infty$. Summarizing we have shown that for $1 \le k \le N$,

(109.3) $$x_k(t) = -2t\lambda_k + \beta_k^+ + o(1) \qquad \text{as } t \to +\infty$$

(109.4) $$x_k(t) = -2t\lambda_{N-k+1} + \beta_k^- + o(1) \qquad \text{as } t \to -\infty$$

where

(109.5) $$\beta_k^+ = \frac{1}{N}\sum_{j=1}^{N} x_j(0) - \frac{2}{N}\sum_{j=1}^{N}\log\left(\frac{u_k(1)}{u_j(1)}\,\frac{\prod_{l=1}^{k-1}(2\lambda_l - 2\lambda_k)}{\prod_{l=1}^{j-1}(2\lambda_l - 2\lambda_j)}\right)$$

and

(109.6) $$\beta_k^- = \frac{1}{N}\sum_{j=1}^{N} x_j(0) - \frac{2}{N}\sum_{j=1}^{N}\log\left(\frac{u_{N-k+1}(1)}{u_{N-j+1}(1)}\,\frac{\prod_{l=N-k+2}^{N}(2\lambda_{N-k+1} - 2\lambda_l)}{\prod_{l=N-j+2}^{N}(2\lambda_{N-j+1} - 2\lambda_l)}\right)$$

We now compute the <u>phase</u> shifts $\beta_k^+ - \beta_{N-k+1}^-$

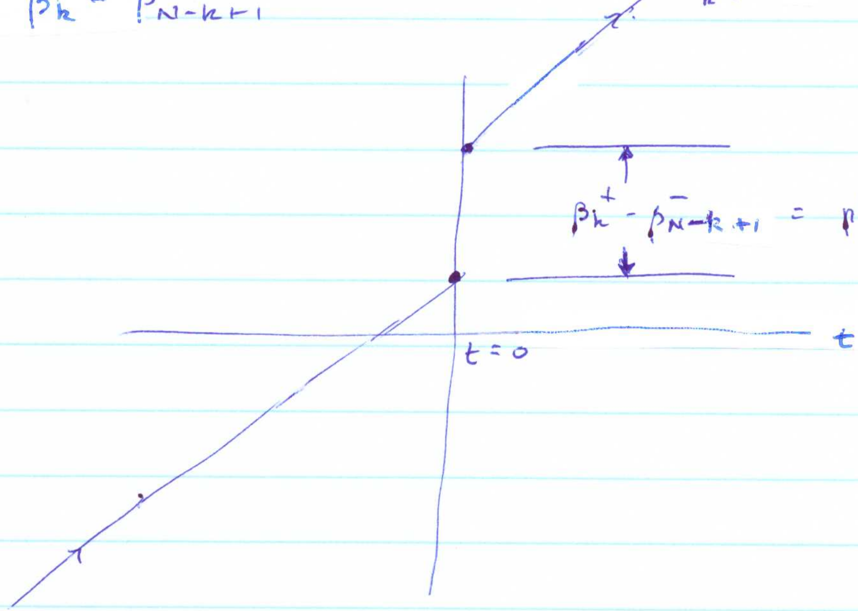
Consider $x_k(t) \sim -2\lambda_k t$ as $t \to +\infty$

$x_{N-k+1}(t) \sim -2\lambda_k t$ as $t \to -\infty$

Thus particle $x_{N-k+1}(t)$ travelling with velocity $-2\lambda_k t$ as $t \to -\infty$, "transfers" its velocity to $x_k(t)$ after collision as $t \to +\infty$. This is why TC phase shift is defined

as $\beta_k^+ - \beta_{N-k+1}^-$

$x_k(t) \sim -2t\lambda_k + \beta_k^+$, $t \to +\infty$



$x_{N-k+1}(t) \sim -2t\lambda_k + \beta_{N-k+1}^-$, $t \to -\infty$

We have

$$\beta_k^+ - \beta_{N-k+1}^- = -\frac{2}{N}\sum_{j=1}^{N} \log\left[\frac{u_k(1)}{u_j(1)}\,\frac{\prod_{l=1}^{k-1}(2\lambda_l - 2\lambda_k)}{\prod_{l=1}^{j-1}(2\lambda_l - 2\lambda_j)}\;\frac{u_j(1)}{u_k(1)}\,\frac{\prod_{l=j+1}^{N}(2\lambda_j - 2\lambda_l)}{\prod_{l=k+1}^{N}(2\lambda_k - 2\lambda_l)}\right]$$

$$= -\frac{1}{N}\sum_{j=1}^{N}\left[\sum_{l=1}^{k-1}\log(2\lambda_l - 2\lambda_k)^2 - \sum_{l=k+1}^{N}\log(2\lambda_k - 2\lambda_l)^2\right]$$

$$-\frac{1}{N}\sum_{j=1}^{N}\sum_{l=j+1}^{N}\log(2\lambda_j - 2\lambda_l)^2 + \frac{1}{N}\sum_{j=1}^{N}\sum_{l=1}^{j-1}\log(2\lambda_l - 2\lambda_j)^2$$

Thus

(110.1) $$\beta_k^+ - \beta_{N-k+1}^- = \sum_{l \neq k} \phi_{lk}, \quad 1 \le k \le N$$

where

(111.1)
$$\phi_{lk} = \log(2\lambda_l - 2\lambda_k)^2 \quad , \quad l > k$$
$$= -\log(2\lambda_l - 2\lambda_k)^2 \quad , \quad l < k .$$

Here we have noted (cf. (106.01)) that $\sum_{l=N+1}^{N} \log(2\lambda_{N+1} - 2\lambda_l)^2 \equiv 0 \equiv \sum_{l=1}^{0} \log(2\lambda_l - 2\lambda_1)$

and so
$$\sum_{j=1}^{N} \sum_{l=j+1}^{N} \log(2\lambda_j - 2\lambda_l)^2 = \sum_{1 \le j < l \le N} \log(2\lambda_j - 2\lambda_l)^2$$
$$= \sum_{l=2}^{N} \sum_{j=1}^{l-1} \log(2\lambda_j - 2\lambda_l)^2$$
$$= \sum_{j=2}^{N} \sum_{l=1}^{j-1} \log(2\lambda_l - 2\lambda_j)^2$$
$$= \sum_{j=1}^{N} \sum_{l=1}^{j-1} \log(2\lambda_l - 2\lambda_j)^2$$

In particular, we see that for $N = 2$

$$\beta_1^+ - \beta_2^- = \phi_{21} = \log(2\lambda_1 - 2\lambda_2)^2$$
$$\beta_2^+ - \beta_1^- = \phi_{12} = -\log(2\lambda_1 - 2\lambda_2)^2$$

Thus the phase shifts

$$\beta_k^+ - \beta_{N-k+1}^- = \sum_{j \neq k} \phi_{jk} .$$

are just the same as if the interaction takes place two particles at a time.

<u>Remark (111.2)</u>: The remarkable formula (110.1) for the phase shift is due to Moser

(loc. cit), which he derived using a very different, indirect argument.