MA-GY 7043: Linear Algebra II

Eigenvalues and Eigenvectors

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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Outline I

Eigenvalues and Eigenvectors of a Linear Transformation

- Consider a linear transformation $L : V \to V$
- A scalar $\lambda \in \mathbb{F}$ is called an **eigenvalue** of L if there is a nonzero vector $v \in V$ such that any of the following equivalent statements hold:

$$\begin{aligned} L(v) = \lambda v &\iff (L - \lambda I)v = 0 \\ &\iff v \in \ker(L - \lambda I) \end{aligned}$$

- The vector v is called an **eigenvector** for the eigenvalue λ
- $\lambda \in \mathbb{F}$ is an eigenvalue of L if and only if the following equivalent statements hold:

$$\dim(\ker(L - \lambda I)) > 0 \iff \det(L - \lambda I) = 0$$

- The **eigenspace** for an eigenvalue λ of L is

$$E_\lambda(L) = \ker(L - \lambda I) = \{v \in V \ : \ L(v) = \lambda v\}$$

- $E_\lambda(L)$ is a linear subspace of V
- The **geometric multiplicity** of the eigenvalue λ is $\dim(E_\lambda(L))$

Eigenvalues and Eigenvectors of a Square Matrix

- A scalar $\lambda \in \mathbb{F}$ is an **eigenvalue** of a matrix $M \in \mathrm{gl}(n, \mathbb{F})$ if there is a nonzero vector $v \in \mathbb{F}^n$ such that any of the following equivalent statements hold:

$$\begin{aligned} Mv = \lambda v &\iff (M - \lambda I)v = 0 \\ &\iff v \in \ker(M - \lambda I) \end{aligned}$$

- The vector v is called an **eigenvector** for the eigenvalue λ
- $\lambda \in \mathbb{F}$ is an eigenvalue of M if and only if the following equivalent statements hold:

$$\dim(\ker(M - \lambda I)) > 0 \iff \det(M - \lambda I) = 0$$

- The **eigenspace** for an eigenvalue λ is the subspace

$$E_\lambda(M) = \ker(M - \lambda I) = \{v \in \mathbb{V} : \; Mv = \lambda v\}$$

- The **geometric multiplicity** of the eigenvalue λ is $\dim(E_\lambda(M))$

Linear Transformation With Respect to a Basis

- Let $L : V \to V$
- Given a basis

$$E = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix},$$

where exists a matrix M such that for each $1 \leq k \leq n$,

$$L(e_k) = e_j M^j_k$$

- If we denote

$$L(E) = \begin{bmatrix} L(e_1) & \cdots & L(e_n) \end{bmatrix},$$

then

$$L(E) = EM$$

- If $v = Ea = e_j a^j$, then

$$L(v) = L(Ea) = L(E)a = EMa$$

Eigenvalues of Linear Transformation Versus Matrix

- Let $L : V \to V$ be a linear transformation and M be the matrix such that
$$L(E) = EM$$
- If $v = Ea$ is an eigenvector of L for an eigenvalue λ, then
$$\lambda v = L(v) = L(Ea) = L(E)a = EMa$$
and therefore
$$\lambda Ea = EMa$$
- It follows that
$$Ma = \lambda a,$$
- Therefore, $v = Ea$ is an eigenvector of L for the eigenvaelue λ if and only if $a \in \mathbb{F}^n$ is an eigenvector of M for the eigenvalue λ

Linear Transformation With Respect To Different Bases

- Let E and F be bases of V
- There exists a matrix S such that $f_k = e_j S^j_k$, i.e.,

$$F = ES \text{ and } E = FS^{-1}$$

- Given a map $L : V \to V$, there are matrices M and N such that

$$L(E) = EM \text{ and } L(F) = FN$$

- On the other hand,

$$FN = L(F) = L(ES) = L(E)S = EMS = FS^{-1}MS$$

and therefore,

$$N = S^{-1}MS$$

- If $v = Ea = Fb$, then

$$L(v) = EMa = FNb = ESNb = ESS^{-1}MSb = EMSb$$

- Therefore,

$$a = Sb \text{ and } b = S^{-1}a$$

Eigenvectors With Respect to Different Bases

- If $v = Ea = Fb$ is an eigenvector of L for the eigenvalue λ, then λ is an eigenvalue for both M and $N = S^{-1}MS$
- The eigenvector of M for the eigenvalue λ is a
- The eigenvector of N for the eigenvalue λ is $b = S^{-1}a$
- This can be checked directly:

$$Nb = S^{-1}MSb = S^{-1}Ma = S^{-1}(\lambda a) = \lambda S^{-1}a = \lambda b$$

Eigenvalues, and Eigenvectors of Similar Matrices

- Two matrices M and N are called **similar** if there is an invertible matrix S such that
$$N = S^{-1}MS$$
or, equivalently,
$$M = SNS^{-1}$$
- If M and N are similar, then $\det M = \det N$
- M and N have the same eigenvalues, because if a is an eigenvector of M for the eigenvalue λ and $b = S^{-1}a$, then
$$Nb = S^{-1}MSb = S^{-1}Ma = S^{-1}(\lambda a) = \lambda S^{-1}a = \lambda b$$

Characteristic Polynomial of a Matrix

- Let δ^j_k be the element in the j-th row and k-column of the identity matrix, i.e.,

$$\delta^j_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

- Observe that the function $p_M : \mathbb{F} \to \mathbb{F}$ given by

$$\begin{aligned} p_M(x) &= \det(M - xI) \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma)(M - xI)^{\sigma(1)}_1 \cdots (M - xI)^{\sigma(n)}_n \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma)(M^{\sigma(1)}_1 - x\delta^{\sigma(1)}_1) \cdots (M^{\sigma(n)}_n - x\delta^{\sigma(n)}_n) \end{aligned}$$

  is a polynomial in x of degree n
- p_M is the **characteristic polynomial** of M
- x is a root of p_M if and only if it is an eigenvalue for M

Characteristic Polynomial of a Linear Transformation

- Let $L : V \to V$ be a linear transformation
- Define $p_L : \mathbb{F} \to \mathbb{F}$ by

$$p_L(x) = \det(L - xI)$$

- If E is a basis and $L(E) = EM$, then

$$(L - xI)(E) = E(M - xI)$$

  and therefore

$$p_L(x) = \det(L - xI) = \det(M - xI) = p_M(x)$$

- It follows that p_L is a polynomial of degree n

Similar Matrices Have the Same Characteristic Polynomial

- **Proof 1:** If $L(E) = EM$ and $L(F) = FN$, then

$$p_M(x) = p_L(x) = p_N(x)$$

- **Proof 2:** If $M = SNS^{-1}$, then

$$M - xI = S(N - xI)S^{-1}$$

and therefore

$$p_M(x) = \det(M - xI) = \det(S(N - xI)S^{-1} = \det(N - xI) = p_N(x)$$

Examples

- Let
$$Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
- $Zv = 0v$ for any $v \in \mathbb{R}^2$ and therefore 0 is the only eigenvalue
- Any nonzero vector $v \in \mathbb{R}^2$ is an eigenvector
- The characteristic polynomial is
$$p_Z(x) = \det(Z - xI) = \det\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = -x^2$$

Examples

- If $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, then $D \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} av^1 \\ bv^2 \end{bmatrix}$
- If $x = a = b$, then the only eigenvalue is x
  - Every $v \in \mathbb{R}^2$ is an eigenvector
- If $a \neq b$, then the only eigenvalues are a and b
  - The eigenvectors for the eigenvalue a are

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ x \in \mathbb{F} \backslash \{0\}$$

  - The eigenvectors for the eigenvalue b are

$$\begin{bmatrix} 0 \\ x \end{bmatrix} = x \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ x \in \mathbb{F} \backslash \{0\}$$

- The characteristic polynomial is

$$p_D(x) = \det(D - xI) = x \begin{bmatrix} a - x & 0 \\ 0 & b - x \end{bmatrix} = (a - x)(b - x)$$

Examples

- If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $A \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} v^2 \\ v^1 \end{bmatrix}$
- The only eigenvalues are $1, -1$
- The eigenvectors for the eigenvalue 1 are

$$\begin{bmatrix} x \\ x \end{bmatrix}, \; x \in \mathbb{F} \backslash \{0\}$$

- The eigenvectors for the eigenvalue -1 are

$$\begin{bmatrix} x \\ -x \end{bmatrix}, \; x \in \mathbb{F} \backslash \{0\}$$

- The characteristic polynomial is

$$p_A(x) = \det \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - x \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} \right) = \det \left(\begin{bmatrix} -x & 1 \\ 1 & x \end{bmatrix} \right) = 1 - x^2$$

Examples

- If $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then $B \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} -v^2 \\ v^1 \end{bmatrix}$
- There are no real eigenvalues
- The complex eigenvalues are $i, -i$
- The eigenvectors for the eigenvalue i are

$$\begin{bmatrix} ix \\ -x \end{bmatrix} = x \begin{bmatrix} i \\ 1 \end{bmatrix}, \ x \in \mathbb{F} \setminus \{0\}$$

- The eigenvectors for the eigenvalue $-i$ are

$$\begin{bmatrix} x \\ ix \end{bmatrix} = x \begin{bmatrix} 1 \\ i \end{bmatrix}, \ x \in \mathbb{F} \setminus \{0\}$$

- The characteristic polynomial is

$$\begin{aligned} p_B(x) &= \det(B - xI) \\ &= \det\left(\begin{bmatrix} -x & -1 \\ 1 & -x \end{bmatrix}\right) \\ &= 1 + x^2 \end{aligned}$$

Complex Versus Real Eigenvalues

- If an $n - by - n$ matrix contains only real entries, it can have anywhere from 0 to n eigenvalues
- A polynomial with complex coefficients

$$p(x) = a_0 + a_1 x + \cdots a_n x^n,$$

where $a_n \neq 0$ with complex coefficients can always be factored into n linear factors

$$p(x) = a_n(r_1 - x) \cdots (r_n - x)$$

- A complex matrix A always has anywhere from 1 to n eigenvalues, where an eigenvalue might appear more than once in the factorization of p_A
- The **algebraic multiplicity** of an eigenvalue λ is the number of linear factors equal to $(\lambda - x)$ in p_A

Examples

- Let $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
- The eigenvalues of D are $-2, 3$
- The characteristic polynomial of D is

$$p_D(\lambda) = (x-3)(x+2)(x-3) = (x-3)^2(x+2)$$

- The eigenvalue 3 has multiplicity 2, and the eigenvalue 2 has multiplicity 1
- The eigenvectors for the eigenvalue -2 are

$$\begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} = x \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; x \in \mathbb{F} \backslash \{0\}$$

- The eigenvectors for the eigenvalue 3 are

$$\begin{bmatrix} x^1 \\ 0 \\ x^2 \end{bmatrix} = x^1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; x \in \mathbb{F} \backslash \{0\}$$

Examples

- Let $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
- The characteristic polynomial of M is

$$p_M(\lambda) = \det(M - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2$$

- The only eigenvalue is 1 with multiplicity 2
- Since

$$M \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} v^1 \\ v^1 + v^2 \end{bmatrix},$$

  the eigenvectors of the eigenvalue 1 are

$$\begin{bmatrix} 0 \\ x \end{bmatrix} = x \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Diagonal Matrices

- An n-by-n matrix M is **diagonal** if
$$M^j_k = 0 \text{ if } j \neq k$$
- In particular, the k-th column of M is
$$C_k = Me_k = M^k_k e_k \text{ (no sum over } k\text{)},$$
where $(e_1, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$
- The determinant of M is, by multilinearity,
$$\begin{aligned} D(C_1, \dots, C_n) &= D(M^1_1 e_1, M^2_2 e_2, \dots, M^n_n e_n) \\ &= (M^1_1 \cdots M^n_n) D(e_1, \dots, e_n) \\ &= M^1_1 \cdots M^n_n \end{aligned}$$
- Since $M - \lambda I$ is also diagonal, it follows that the characteristic polynomial of M is
$$p_M(\lambda) = \det(M - \lambda I) = (M^1_1 - \lambda) \cdots (M^n_n - \lambda)$$
- The diagonal elements of M are its eigenvalues

Triangular Matrices

- An n-by-n matrix M is **upper triangular** if it is of the form

$$M = \begin{bmatrix} M_1^1 & M_2^1 & \cdots & M_{n-1}^1 & M_n^1 \\ 0 & M_2^2 & \cdots & M_{n-1}^2 & M_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{n-1}^{n-1} & M_n^{n-1} \\ 0 & 0 & \cdots & 0 & M_n^n \end{bmatrix}$$

- I.e., $M_k^j = 0$ if $j > k$
- An n-by-n matrix M is **lower triangular** if it is of the form

$$M = \begin{bmatrix} M_1^1 & 0 & \cdots & 0 & 0 \\ M_1^2 & M_2^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ M_1^{n-1} & M_2^{n-1} & \cdots & M_{n-1}^{n-1} & 0 \\ M_1^n & M_2^n & \cdots & M_{n-1}^n & M_n^n \end{bmatrix}$$

- I.e., $M_k^j = 0$ if $j < k$

Columns of an Upper Triangular Matrix

- Let M be an upper triangular matrix and consider the matrix $T = M - \lambda I$
- T is itself an upper triangular matrix
- Choose a value of $\lambda \in \mathbb{F}$ such that every element on the diagonal of T is nonzero
- Let $(e_1, \ldots, e_n)$ be the standard basis of $\mathbb{R}^n$
- Let $(C_1, \ldots, C_n)$ be the columns of T
- By assumption, $C_1^1, C_2^2, \cdots, C_n^n$ are all nonzero

Columns of Upper Triangular Matrix (Part 2)

- Each column can therefore be written as

$$C_k = C_k^k \hat{C}_k,$$

where

$$\hat{C}_k = \begin{bmatrix} \hat{C}_k^1 \\ \vdots \\ \hat{C}_k^{k-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } \hat{C}_k^j = \frac{C_k^j}{C_k^k}, \text{ for each } 1 \leq j, k \leq n$$

Determinant of Upper Triangular Matrix (Part 1)

- Let $(C_1, \ldots, C_n)$ be the columns of T and recall that the determinant of T is

$$\det(T) = D(C_1, \ldots, C_n)$$

where $D \in \Lambda^n V^*$ satisfies $D(e_1, \ldots, e_n) = 1$

- By the multilinearity of D,

$$\begin{aligned} D(C_1, \ldots, C_n) &= D(C_1^1 \hat{C}_1, C_2^2 \hat{C}_2, \ldots, C_n^n \hat{C}_n) \\ &= (C_1^1 C_2^2 \cdots C_n^n) D(\hat{C}_1, \ldots, \hat{C}_n) \end{aligned}$$

Determinant of Upper Triangular Matrix (Part 2)

- Since T is upper triangular, its columns are of the form

$$\begin{aligned} C_1 &= C_1^1 e_1 \\ C_2 &= C_2^1 e_1 + C_2^2 e_2 \\ C_3 &= C_3^1 e_1 + C_3^2 e_2 + C_3^3 e_3 \\ &\vdots \\ C_n &= C_n^1 e_1 + C_n^2 e_2 + C_n^3 e_3 + \cdots + C_n^n e_n \end{aligned}$$

- Similarly,

$$\begin{aligned} \hat{C}_1 &= e_1 \\ \hat{C}_2 &= \hat{C}_2^1 e_1 + e_2 \\ \hat{C}_3 &= \hat{C}_3^1 e_1 + \hat{C}_3^2 e_2 + e_3 \\ &\vdots \\ \hat{C}_n &= \hat{C}_n^1 e_1 + \hat{C}_n^2 e_2 + \hat{C}_n^3 e_3 + \cdots + \hat{C}_n^{n-1} e_{n-1} + e_n \end{aligned}$$

Determinant of Upper Triangular Matrix (Part 3)

- Therefore,

$$
\begin{aligned}
& D(\hat{C}_1, \ldots, \hat{C}_n) \\
& = D(e_1, \hat{C}_2, \ldots, \hat{C}_n) \\
& = D(e_1, \hat{C}_2^1 e_1 + e_2, \hat{C}_3^1 e_1 + \hat{C}_3^2 e_2 + e_3, \ldots, \hat{C}^1 e_1 + \cdots + e_n) \\
& = D(e_1, e_2, \hat{C}_3^2 e_2 + e_3, \ldots, \hat{C}_n^2 e_2 + \cdots + e_n) \\
& = D(e_1, e_2, e_3, \ldots, \hat{C}_n^3 e_3 + \cdots + \cdots + e_n) \\
& \vdots \qquad \vdots \\
& = D(e_1, e_2, \ldots, e_n) \\
& = 1
\end{aligned}
$$

Characteristic Polynomial and Determinant of Triangular Matrix

- It follows that if λ is not equal to any of $C_1^1, \cdots, C_n^n$,

$$\begin{aligned} p_M(\lambda) &= \det(T) \\ &= D(C_1, \ldots, C_n) \\ &= C_1^1 C_2^2 \cdots C_n^n D(\hat{C}_1, \ldots, \hat{C}_n) \\ &= C_1^1 C_2^2 \cdots C_n^n \\ &= (M_1^1 - \lambda I) \cdots (M_n^n - \lambda I) \end{aligned}$$

- Therefore, the polynomial

$$r(\lambda) = p_M(\lambda) - (M_1^1 - \lambda I) \cdots (M_n^n - \lambda I)$$

  has infinitely many roots
- This implies that r is the zero polynomial
- The characteristic polynomial of an upper triangular matrix M is

$$p_M(\lambda) = (M_1^1 - \lambda I) \cdots (M_n^n - \lambda I)$$

- In particular, $\det(M) = p_M(0) = M_1^1 \cdots M_n^n$

Diagonal Linear Transformation

- Let $\dim V = n$
- Let $L : V \to V$ be a linear transformation
- Suppose L has n linearly independent eigenvectors $e_1, \ldots, e_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$
- Then with respect to the basis $E = (e_1, \ldots, e_n)$,

$$L(e_k) = e_k \lambda_k$$

- Equivalently,

$$\begin{bmatrix} L(e_1) & \cdots & L(e_n) \end{bmatrix} = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Diagonal Linear Transformation

- Conversely, suppose $L : V \to V$ is a linear transformation and E is a basis such that
$$L(E) = ED,$$
where D is a diagonal matrix
- Then
$$L(e_k) = e_j D^j_k = e_k D^k_k$$
- Therefore, L has eigenvalues $D^1_1, \ldots, D^n_n$ with eigenvectors $e_1, \ldots, e_n$ respectively

Diagonalizable Linear Transformation

- Let $L : V \to V$ be a diagonal linear transformation
- If E is a basis of eigenvectors, then

$$L(E) = ED,$$

  where D is a diagonal matrix
- Given any basis F, there is an invertible matrix M such that

$$F = EM$$

  and vice versa
- There is a matrix A such that

$$L(F) = FA$$

- Therefore,

$$ED = L(E) = L(FM^{-1}) = L(F)M^{-1} = FAM^{-1} = EMAM^{-1}$$

- I.e., M and D are similar

Diagonalizable Linear Transformation and Matrix

- A linear transformation $L : V \to V$ is **diagonalizable** if any of the following equivalent conditions hold:
  - There exists a basis of V consisting of eigenvectors
  - There exists a basis E such that $L(E) = ED$, where D is a diagonal matrix
  - Given any basis F and matrix A such that

    $$L(F) = FA,$$

    A is similar to a diagonal matrix
  - A matrix A is **diagonalizable** if it is similar to a diagonal matrix

Linear Transformation With Distinct Eigenvalues

- Let $\dim(V) = n$ and $L : V \to V$ be a linear transformation with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, i.e.,

$$j \neq k \implies \lambda_j \neq \lambda_k$$

- Let $v_1, \ldots, v_n$ be eigenvectors of $\lambda_1, \ldots, \lambda_n$ respectively
- Suppose $v_1, \ldots, v_{k-1}$ are linearly independent
- If $a^1 v_1 + \cdots + a^k v_k w = 0$, then

$$\begin{aligned} 0 &= (L - \lambda_k I)(a^1 v_1 + \cdots + a^k v_k) \\ &= a^1(Lv_1) - \lambda_k v_1) + \cdots + a^k(L(v_k) - \lambda_k v_k) \\ &= a^1(\lambda_1 - \lambda_k)v_1 + \cdots + a^k(\lambda_k - \lambda_k)v_k \\ &= a^1(\lambda_1 - \lambda_k)v_1 + \cdots + a^{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1}, \end{aligned}$$

- Therefore, $a^1(\lambda_1 - \lambda_k) = \cdots = a^{k-1}(\lambda_{k-1} - \lambda_k) = 0$

Linear Transformation With Distinct Eigenvalues

- Since $v_1, \ldots, v_{k-1}$ are linearly independent, it follows that

$$a^1(\lambda_1 - \lambda_k) = \cdots = a^{k-1}(\lambda_{k-1} - \lambda_k) = 0$$

- Since the eigenvalues are distinct, this implies that

$$a^1 = \cdots = a^{k-1} = 0$$

- By assumption, $a^1 v_1 + \cdots + a^k v_k w = 0$ and therefore $a^k = 0$
- It follows by induction that $v_1, \ldots, v_n$ form a basis of V
- Therefore, L is diagonalizable
- **Conclusion:** Any linear transformation with n distinct eigenvalues is diagonalizable

Direct Sum of Subspaces

- Let $V_1, \ldots, V_k$ be subspaces of V
- $\{V_1, \ldots, V_k\}$ is a **linearly independent** set of subspaces if for any nonzero vectors

$$v_1 \in V_1,\ v_2 \in V_2, \ldots, v_k \in V_k$$

  are linearly independent
- Equivalently, $\{V_1, \ldots, V_k\}$ is linearly independent if for any $v_1 \in V_1, \ldots, v_k \in V_k$,

$$v_1 + v_2 + \cdots + v_k = 0 \implies v_1 = v_2 = \cdots = v_k$$

- Equivalently, $\{V_1, \ldots, V_k\}$ is linearly independent if for any $v_1, w_1 \in V_1, \ldots, v_k, w_k \in V_k$,

$$v_1 + v_2 + \cdots + v_k = w_1 + w_2 + \cdots + w_k \implies v_1 = w_1, \ldots, v_k = w_k$$

- If $\{V_1, V_2, \ldots, V_k\}$ is linearly independent, then their **direct sum** is defined to be

$$V_1 \oplus V_2 \oplus \cdots \oplus V_k = \text{span}(V_1 \cup V_2 \cup \cdots \cup V_k)$$

Examples

- $\{S_1, S_2\}$, where $S_1, S_2 \subset \mathbb{F}^3$ are given by

$$S_1 = \text{span}(e_1)$$
$$S_2 = \text{span}(e_2),$$

  is linearly independent
- If $\{v_1, \dots, v_k\}$ is linearly independent and

$$\forall 1 \leq j \leq k,\ \ V_j = \text{span}(v_j),$$

  then $\{V_1, \dots, V_k\}$ is a linearly independent set of subspaces
- If (e_1, e_2, e_3, e_4) is a basis of V and

$$S = \text{span}(e_1, e_2, e_3),\ \ T = \text{span}(e_4),$$

  then $V = S \oplus T$

Eigenspaces of Distinct Eigenvalues are Linearly Independent (Part 1)

- If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of $L : V \to V$, then their eigenspaces $E_{\lambda_1}, \ldots, E_{\lambda_k}$ are linearly independent
- Prove by induction that for any $1 \leq j \leq k$,

$$v_1 + \cdots + v_j = 0 \implies v_1 = \cdots = v_j = 0$$

- This holds for $j = 1$
- Inductive step: Assume that it holds for $1 \leq j < k$ and prove it holds for $j + 1$

Eigenspaces of Distinct Eigenvalues are Linearly Independent (Part 2)

- Suppose $v_1 \in E_{\lambda_1}, \ldots, v_{j+1} \in E_{\lambda_{j+1}}$ satisfy

$$v_1 + \cdots + v_{j+1} = 0 \tag{1}$$

- It follows that

$$\begin{aligned} 0 &= (L - \lambda_{j+1} I)(v_1 + \cdots + v_{j+1}) \\ &= (\lambda_1 - \lambda_{j+1})v_1 + \cdots + (\lambda_j - \lambda_{j+1})v_j \end{aligned}$$

- By the inductive assumption,

$$(\lambda_1 - \lambda_{j+1})v_1 = \cdots = (\lambda_j - \lambda_{j+1})v_j = 0$$

- Since $\lambda_i - \lambda_{j+1} \neq 0$ for each $1 \leq i \leq j$,

$$v_1 = \cdots = v_j = 0$$

- By (1), it follows that $v_{j+1} = 0$

Eigenspaces of Distinct Eigenvalues are Linearly Independent (Part 3)

- By induction,

$$v_1 + \cdots + v_k = 0 \implies v_1 = \cdots = v_k = 0$$

- This implies that $E_{\lambda_1}, \ldots, E_{\lambda_k}$ are linearly independent

Diagonalizability of a Linear Transformation (Part 1)

- Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of $L : V \to V$
- L is diagonalizable if and only if

$$\dim(E_{\lambda_1}) + \cdots + \dim(E_{\lambda_k}) = \dim V$$

- Let $n_0 = 0$ and, for $1 \leq j \leq k$, let

$$n_j = \dim(E_{\lambda_j})$$
$$N_j = n_1 + \cdots + n_j$$

- For each $1 \leq j \leq k$, let

$$(v_{N_{j-1}+1}, \cdots, v_{N_j})$$

be a basis of E_{λ_j}

Diagonalizability of a Linear Transformation (Part 2)

- Suppose
$$a^1 v_1 + \cdots + a^n v_n = 0,$$
- For each $1 \leq j \leq k$, let
$$w_j = a^{N_{j-1}+1} v_{N_{j-1}} + \cdots + a^{N_j} v_{N_j} \in E_{\lambda_j}$$
- Since $w_1 + \cdots + w_k = 0$, it follows that
$$w_1 = \cdots = w_k = 0$$
- For each $1 \leq j \leq k$,
$$0 = w_j = a^{N_{j-1}+1} v_{N_{j-1}} + \cdots + a^{N_j} v_{N_j},$$
which implies $a^{N_{j-1}+1} = \cdots = a^{N_j} = 0$
- Therefore, $(v_1, \ldots, v_n)$ is a basis of V
- L is diagonal with respect to this basis