MA-GY 7043: Linear Algebra II

Self-Adjoint Transformations
Singular Value Decomposition

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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Outline I

Self-Adjoint Maps and Symmetric Matrices

- Given a Hermitian vector space V, a linear map $L : V \to V$ is **self-adjoint** if
$$L^* = L$$
- A complex matrix M is **Hermitian** if
$$M^* = M$$

Eigenvalues of a Self-Adjoint Map are Real

- Let $L : V \to V$ be a hermitian linear map with basis $(e_1, \cdots, e_n)$
- If v is an eigenvector of L with eigenvalue λ, then

$$
\begin{aligned}
\lambda\|v\|^2 &= (L(v), v) \\
&= (v, L(v)) \\
&= \overline{(L(v), v)} \\
&= \overline{\lambda\|v\|^2} \\
&= \overline{\lambda}\|v\|^2
\end{aligned}
$$

Eigenspaces of a Self-Adjoint Map are Orthogonal

- Suppose λ, μ are two different eigenvalues of a self-adjoint operator $L : V \to V$ with eigenvectors v, w respectively
- It follows that

$$\begin{aligned} 0 &= (L(v), w) - (v, L(w)) \\ &= (\lambda v, w) - (v, \mu w) \\ &= (\lambda - \mu)(v, w) \text{ since } \mu \in \mathbb{R} \end{aligned}$$

- Since $\lambda - \mu \neq 0$, it follows that $(v, w) = 0$

Self-Adjoint Map Has Unitary Basis of Eigenvectors

- **Theorem.** Given a self-adjoint map $L : V \to V$, there exists a unitary basis of eigenvectors
- **Corollary.** Given a Hermitian matrix M, there exists a unitary matrix $U \in U(n)$ and real diagonal matrix D such that

$$M = UDU^*,$$

Proof of Theorem

- Given a linear map $L : V \to V$, by the Schur decomposition, there exists a unitary basis $(u_1, \dots, u_n)$ such that
$$L(u_k) = u_k M_k^k + \cdots u_n M_k^n, \text{ for each } 1 \leq k \leq n$$
- Equivalently, for any $1 \leq k \leq n$ and $1 \leq j < k$,
$$(L(u_k), u_j) = 0$$
- If L is self-adjoint, then for any $1 \leq k \leq n$ and $1 \leq j < k$,
$$0 = (L(u_k), u_j) = (u_k, L^*(u_j)) = (u_k, L(u_j)) = \overline{(L(u_j), u_k)},$$
which implies $(L(u_j), u_k) = 0$
- Therefore, for any $1 \leq k \leq n$ and $1 \leq j < k$,
$$(L(u_j), u_k) = 0$$
- It follows that for each $1 \leq k \leq n$,
$$L(u_k) = M_k^k u_k$$

Proof of Corollary

- A self-adjoint matrix M defines a self-adjoint linear map

$$M : \mathbb{F}^n \to \mathbb{F}^n$$

- By the Theorem, there exists a unitary basis $(u_1, \ldots, u_n)$ of eigenvectors of M with eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively
- Let U be the matrix whose columns are $u_1, \ldots, u_n$,

$$U = \left[\begin{array}{c|c|c} u_1 & \cdots & u_n \end{array}\right]$$

- If $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{F}^n$, then for $1 \leq k \leq n$,

$$Ue_k = u_k$$

- Therefore,

$$U^* MUe_k = U^* Mu_k = U^*(\lambda_k u_k) = \lambda e_k$$

- It follows that $U^* MU$ is a diagonal matrix with $\lambda_1, \ldots, \lambda_n$ as the diagonal entries

Direct Proof of Corollary

- Given a square matrix M, by the Schur decomposition, there exists a unitary matrix U such that
$$M = UTU^*,$$
where T is upper triangular
- If M is self-adjoint, then
$$UTU^* = M = M^* = (U^*)^* T^* U^* = UT^* U^*$$
- Therefore,
$$T = U^* UTU^* U = U^*(UTU^*)U = U^*(UT^* U^*)U = T^*,$$
which implies T is self-adjoint
- Since T is upper triangular,
$$T^j_k = 0, \text{ if } j < k \leq n$$
- Since $T^* = T$,
$$T^k_j = (T^*)^k_j = \bar{T}^j_k = 0, \text{ if } j < k \leq n$$
- It follows that T is diagonal

Positive Definite Self-Adjoint Maps

- Let V be an inner product space
- A self-ajoint map $L : V \to V$ is **positive definite** if for any $v \neq 0$,
$$(L(v), v) > 0$$
- If L is a positive definite self-adjoint map, we write $L > 0$
- $L > 0$ if and only if the eigenvalues of L are all positive
- We write $L \geq 0$ if the eigenvalues of L are all nonnegative
- The function $Q : V \times V \to \mathbb{F}$ given by
$$Q(v, w) = (L(v), v)$$
is an inner product

Powers and Roots of a Positive Definite Self-Adjoint Map

- Let L be a self-adjoint map such that $L \geq 0$ and $(u_1, \ldots, u_n)$ be a unitary basis of eigenvectors
- There is a unique self-adjoint map $\sqrt{L} \geq 0$ such that

$$\sqrt{L} \geq 0 \text{ and } \sqrt{L} \circ \sqrt{L} = L$$

  - Let

$$\sqrt{L}(u_k) = \sqrt{\lambda_k} u_k$$

- If k is a nonnegative integer and $L \geq 0$, then there is a unique self-adjoint map $L^{1/k}$ such that

$$L^{1/k} \geq 0 \text{ and } (L^{1/k}) = L$$

  - Let

$$L^{1/k}(u_j) = \lambda_j^{1/k} u_j$$

Singular Values of a Linear Map

- Let X and Y be inner product spaces (not necessarily of the same dimension)
- Let $L : X \to Y$ be *any* linear map
- The map $L^*L : X \to X$ is self-adjoint, because for any $x_1, x_2 \in X$,

$$(L^*(L(x_1)), x_2)_X = (L(x_1), L(x_2))_Y = (x_1, L^*(L(x_2)))_X$$

- $L^*L \geq 0$, because for any $x \in X$,

$$(L^*L(x), x) = (L(x), L(x)) \geq 0$$

- We can denote $|L| = \sqrt{L^*L}$
- The eigenvalues of $|L|$ are called the **singular values** of L
- Singular values are always real and nonnegative
- Since $\ker L^*L = \ker L$, if $k = \dim \ker L$, then exactly k singular values are zero

Normal Form of a Linear Map (Part 1)

- Let X and Y be complex vector spaces such that $\dim X = m$ and $\dim Y = n$
- Let $L : X \to Y$ be a linear map with rank r
- If $\dim \ker L = k$, then, by Rank Theorem, $r + k = m$
- Let $(u_{r+1}, \dots, u_m)$ be a unitary basis of $\ker L$
- This can be extended to a unitary basis of eigenvectors of $|L| = \sqrt{L^* L}$
- Therefore,
$$|L|(u_j) = s_j u_j, 1 \le j \le m,$$
where $s_1, \dots, s_m$ are the singular values of L
- Observe that
$$\begin{aligned} s_1, \dots, s_r &> 0 \\ s_{r+1} = \cdots = s_m &= 0 \end{aligned}$$

Normal Form of a Linear Map (Part 2)

- Let $\tilde{v}_j = L(u_j)$, $1 \le j \le r$
- The set $\{\tilde{v}_1, \ldots, \tilde{v}_r\}$ is linearly independent because if

  $$a^1\tilde{v}_1 + \cdots + a^k\tilde{v}_r = 0,$$

  then

  $$L(a^1u_1 + \cdots + a^ku_r) = a^1\tilde{v}_1 + \cdots + a^k\tilde{v}_r = 0$$

  which implies that $a^1u_1 + \cdots + a^ku_r \in \ker L$ and therefore

  $$a^1u_1 + \cdots + a^ku_r = b^{r+1}u_{r+1} + \cdots + b^mu_m,$$

  which implies that $a^1 = \cdots = a^k = b^{r+1} = \cdots = b^m = 0$
- Moreover, if $1 \le i, j \le k$, then $s_j \ne 0$ and therefore

  $$(\tilde{v}_i, \tilde{v}_j) = (L(u_i), L(u_j)) = (u_i, (L^*L)(u_j)) = s_j^2(u_i, u_j) = s_j^2\delta_{ij}$$

Normal Form for a Linear Map (Part 3)

- If
$$v_j = s_j^{-1}\tilde{v}_j = s_j^{-1}L(u_j),\ 1 \le j \le k,$$
then $(v_1, \ldots, v_r)$ is a unitary basis of $L(X) \subset Y$ and therefore a unitary set in Y
- This can be extend, using Gram-Schmidt, to a unitary basis $(v_1, \ldots, v_n)$ of Y
- Therefore, there is a unitary basis $(u_1, \cdots, u_m)$ of X and a unitary basis $(v_1, \ldots, v_n)$ of Y such that
$$L(u_j) = \begin{cases} s_j v_j & \text{if } 1 \le j \le \dim\ker L \\ 0 & \text{if } j > \dim\ker L \end{cases}$$

Normal Form for a Linear Map (Part 4)

- $$\left[\begin{array}{ccc|ccc} L(u_1) & \cdots & L(u_r) & L(u_{r+1}) & \cdots & L(u_m) \end{array}\right]$$
  $$= \left[\begin{array}{ccc|ccc} v_1 & \cdots & v_r & v_{r+1} & \cdots & v_n \end{array}\right] \left[\begin{array}{c|c} D & 0_{r\times(m-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{array}\right],$$

  where D is the r-by-r diagonal matrix such that

  $$D^i_j = \begin{cases} s_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- $$L(a^1u_1 + \cdots + a^m u_m)$$
  $$= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \left[\begin{array}{c|c} D & 0_{r\times(m-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{array}\right] \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix}$$

Singular Value Decomposition of Linear Map (Part 1)

- Let X, Y be inner product spaces, where $m = \dim(X)$ and $n = \dim(Y)$
- Let $L : X \to Y$ be a linear map with rank r
- Recall that
$$|L| = (L^* L)^{1/2}$$
is a self-adjoint map with nonnegative eigenvalues
- The singular values $s_1, \ldots, s_r$ of L are defined to be the nonzero eigenvalues of $|L|$
- Let $\Sigma : \mathbb{R}^m \to \mathbb{R}^m$ be the m-by-m diagonal matrix such that for each $1 \leq k \leq m$,
$$\Sigma(\epsilon_k) = \begin{cases} s_k e_k & \text{if } 1 \leq k \leq r \\ 0 & \text{if } r+1 \leq k \leq m \end{cases}$$

Singular Value Decomposition of Linear Map (Part 2)

- There exists a unitary basis $(e_1, \ldots, e_m)$ of X such that

$$|L|(e_k) = \begin{cases} s_k e_k & \text{if } 1 \leq k \leq r \\ 0 & \text{if } r+1 \leq k \leq m \end{cases}$$

- Let $V : \mathbb{R}^m \to X$ be the map such that for each $1 \leq k \leq m$,

$$V(\epsilon_k) = e_k$$

- There exists a unitary basis $(f_1, \ldots, f_n)$ of Y such that for each $1 \leq j \leq n$,

$$f_j = s_j^{-1} L(e_j),\ 1 \leq j \leq n$$

- Let $W : \mathbb{R}^m \to Y$ be the map such that for each $1 \leq j \leq n$,

$$W(\epsilon_j) = f_j$$

- Then

$$L = W \circ \Sigma \circ V^* \text{ and } |L| = V \Sigma V^*$$

Singular Value Decomposition of a Complex Matrix

- If M is an n-by-m complex matrix with rank r and whose positive singular values are $s_1, \ldots, s_r$, then there are unitary matrices $P \in U(m)$ and $Q \in U(n)$ such that

$$M = QSP,$$

where

$$S = \left[\begin{array}{c|c} D & 0_{r\times(m-r)} \\ \hline 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{array}\right]$$

and D is the r-by-r diagonal matrix such that

$$D^i_j = \begin{cases} s_j & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Polar Decomposition of Linear Map

- Let X and Y be inner product spaces such that $\dim(X) = \dim(Y)$
- Consider a linear map
$$L : X \to Y$$
- Then there exists a unitary map $U : X \to Y$ such that
$$L = U|L|$$
- Proof: By the singular value decomposition of L,
$$L = W\Sigma V^* = (WV^*)V\Sigma V^* = U|L|$$