MA-GY 7043: Linear Algebra II

Quasisolutions to System of Linear Equations
Moore-Penrose Quasi-Inverse Operator
Unit Ball of Inner Product Space
Operator and Frobenius Norms of Linear Map
Condition Number of Linear Map

Deane Yang

Courant Institute of Mathematical Sciences
New York University

March 13, 2025

Outline I

System of Linear Equations

- Consider a system of n equations with m unknowns,

$$a_1^1 x^1 + \cdots + a_m^1 x^m = y^1$$
$$\vdots$$
$$a_1^n x^1 + \cdots + a_m^n x^m = y^n$$

- Usually, there is no solution
- And, even if there is a solution, it is usually not unique
- Basic examples
  - 1 equation in 1 unknown

$$3x = 1$$

  - 1 equation in 2 unknowns

$$x + y = 1$$

  - 2 equations in 2 unknowns

$$x + y = 1$$
$$x + y = 2$$

Matrix Equation

- Given $A \in \mathcal{M}_{n\times m}(\mathbb{C})$ and $y \in \mathbb{C}^n$, we want to solve for $x \in \mathbb{C}^m$ such that
$$Ax = y$$
- The matrix A defines a map $A : \mathbb{C}^m \to \mathbb{C}^n$
- There is a solution if and only if $y \in \operatorname{image} A$
- If a solution exists, then it is unique if and only if $\ker A = \{0\}$
- It is possible that $y \notin \operatorname{image} A$, because A and y are from inexact measurements
- Instead, we look for best possible approximation

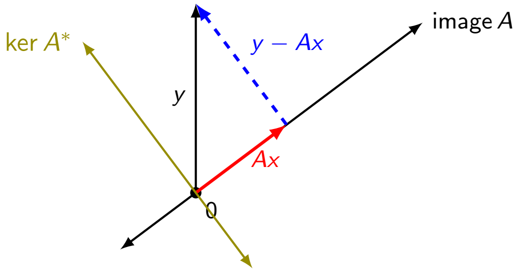
Quasi-Solution with Least Error

- Given $x \in \mathbb{C}^m$, define the error to be

$$\epsilon = L(x) - y \in \mathbb{C}^n$$

- Goal: Solve for x that minimizes the magnitude of the error, $\|\epsilon\|$
- An $x \in X$ that minimizes $\|\epsilon\|$ is called a **quasi-solution**
- If $y \in \text{image}\, L$, then a quasi-solution is a solution
- A quasi-solution need not be unique

Geometric Perspective



- If Ax is closest to y, then
  - $y - Ax$ is orthogonal to image A
- Recall that $(\text{image } A)^\perp = \ker A^*$
- Therefore, A is closest to y if

$$A^*(y - Ax)) = 0$$

or, equivalently,

$$A^*Ax = A^*y$$

Example

- Consider the system of equations

$$\begin{aligned} x + y + z &= 3 \\ x + y &= 3 \\ z &= 3 \end{aligned}$$

- Equivalently,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

- There is no solution

Quasi-Solution

- Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- (x, y, z) is a quasi-solution if

$$A^* A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^* \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

Quasi-Solution Via Row Reduction

- (x, y, z) is a quasi-solution if

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\implies x + y = 2$$

$$z = 2$$

Quasi-Solution Error

- $$\begin{bmatrix} x \\ 2-x \\ 2 \end{bmatrix} \text{ is a quasi-solution to } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$
- The error of the quasi-solution

$$\epsilon = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 2-x \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Error Comparison

- The error for any other (x, y, z) is

$$\epsilon = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} x + y + z - 3 \\ x + y - 3 \\ z - 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} x + y + z - 4 \\ x + y - 2 \\ z - 2 \end{bmatrix}$$

- The error magnitude squared is

$$\epsilon^2 = \left\| \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\|^2 + \left\| \begin{bmatrix} x + y + z - 4 \\ x + y - 2 \\ z - 2 \end{bmatrix} \right\|^2 \geq \left\| \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\|^2$$

Quasi-Solutions of $L(x) = y$

- $L(x)$ is closest to y if
$$L^*L(x) = L^*(y)$$
- For any $y \in Y$, there is always a quasi-solution x, because
$$\text{image}(L^*L) = \text{image}\, L^*$$
  - Recall that $\ker(L^*L) = \ker L$
  - Therefore, since L^*L is self-adjoint,
$$\text{image}(L^*L) = (\ker L^*L)^\perp = (\ker L)^\perp = \text{image}\, L^*$$
- If $v \in \ker L^*L = \ker L$, then $x + v$ is also a solution
- The quasi-solution is unique only if $\ker L = \{0\}$
  - Because the domain and range of L^*L have the same dimension
  - If $\dim X > \dim Y$, this is not possible, because
$$\dim \ker L = \dim X - \dim(\text{image}\, L) \geq \dim X - \dim Y > 0$$

Error Comparison

- A quasi-solution of the equation $L(x) = y$ satisfies

$$L^* L(x) = L^*(y)$$

  and therefore $L^*(L(x) - y) = 0$
- The error of the quasi-solution x is

$$\epsilon = L(x) - y$$

- The error of any $x' \in X$ is

$$\epsilon' = L(x') - y = L(x' - x) + L(x) - y = L(x' - x) + \epsilon$$

- On the other hand,

$$\begin{aligned}(L(x' - x), \epsilon) &= (x' - x, L^*(\epsilon)) \\ &= (x' - x, L^* L(x) - L^*(y)) \\ &= 0\end{aligned}$$

- Therefore, $\|\epsilon'\|^2 = \|\epsilon\|^2 + \|L(x' - x)\|^2$

Quasi-Solution when $L^*L : X \to X$ is Invertible

- If x is a quasi-solution, then

$$L^*L(x) = L^*(y)$$

- If the map $L^*L : X \to X$ is invertible, then the unique quasi-solution is

$$x = (L^*L)^{-1}L^*(y)$$

Solution with Minimal Magnitude

- Suppose $x \in X$ is a solution (not just a quasi-solution) of

$$Ax = y$$

- If $v \in \ker A$, then $x + v$ is also a solution,

$$A(x + v) = y$$

- There is a unique solution x with minimal magnitude

Minimal Magnitude Solution Via Orthogonal Projection

- For any $x' \in X$, there is a unique way to decompose x' into a sum
$$x' = x + (x' - x),$$
where $x \in (\ker A)^\perp$ and $x - x' \in \ker A$
- If x' is a solution to
$$Ax' = y,$$
then
$$Ax = A(x - x') + Ax' = y$$
- If $x_1, x_2 \in (\ker A)^\perp$ are both solutions, then
$$x_1 - x_2 \in (\ker A)^\perp \text{ and } x_1 - x_2 \in \ker A,$$
because
$$A(x_1 - x_2) = Ax_1 - Ax_2 = y - y = 0$$
Therefore, $x_1 - x_2 = 0$

Quasi-Solution with Minimal Magnitude

- A quasi-solution to
$$Ax = y$$
is a solution of
$$A^*Ax = A^*y$$
- There is a unique quasi-solution $x \in (\ker A^*A)^\perp = (\ker A)^\perp$

Example

- The quasi-solutions of the equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

  are

$$\begin{bmatrix} x \\ 2-x \\ 2 \end{bmatrix}, \; x \in \mathbb{C}$$

- The magnitude squared of each quasi-solution is

$$\left\| \begin{bmatrix} x \\ 2-x \\ 2 \end{bmatrix} \right\|^2 = x^2 + (2-x)^2 + 4 = 2((x-1)^2 + 3)$$

- The magnitude is minimized when $x = 1$ and therefore the Moore-Penrose quasi-solution is $(1, 1, 2)$

Moore-Penrose Quasi-Inverse Operator

- Let X and Y be inner product spaces and $L : X \to Y$ be a linear map
- There is a map $L^+ : Y \to X$ such that for any $y \in Y$, $x = L^+(y)$ is the unique quasi-solution with minimal magnitude of the equation
$$L(x) = y$$
- The map L^+ is called the **Moore-Penrose quasi-inverse** of L

Moore-Penrose Quasi-Inverse Operator

- The map
$$L|_{(\ker L)^\perp} : (\ker L)^\perp \to \operatorname{image} L$$
is an isomorphism.
- Let
$$\pi : Y \to \operatorname{image} L$$
be orthogonal projection
- The Moore-Penrose quasi-inverse operator is the map
$$L^+ : Y \to X,$$
given by
$$L^+(y) = \left(L|_{(\ker L)^\perp} \right)^{-1} (\pi(y)) \in (\ker L)^\perp \subset X$$

Quasi-Inverse of Diagonal Matrix

- Let $\Sigma : \mathbb{R}^m \to \mathbb{R}^m$ be the diagonal matrix such that for each $1 \leq k \leq m$,

$$\Sigma(\epsilon_k) = \begin{cases} s_k \epsilon_k & \text{if } 1 \leq k \leq r \\ 0 & \text{if } r+1 \leq k \leq m \end{cases}$$

- Therefore,

$$\Sigma(\epsilon_1 v^1 + \cdots + \epsilon_m v^m) = \epsilon_1 s_1 v^1 + \cdots + \epsilon_r s_r v^r$$

- The quasi-inverse of Σ satisfies the following:

$$\Sigma^+(\epsilon_1 v^1 + \cdots + \epsilon_m v^m) = \epsilon_1 s_1^{-1} v^1 + \cdots + \epsilon_r s_r v^r$$

- In particular,

$$\begin{aligned} \Sigma^+(\Sigma(\epsilon_1 v^1 + \cdots + \epsilon_m v^m)) &= \Sigma^+(\epsilon_1 s_1 v^1 + \cdots + \epsilon_r s_r v^r) \\ &= \epsilon_1 v^1 + \cdots + \epsilon_r v^r \\ &= \pi_r(v), \end{aligned}$$

  where $\pi_r : \mathbb{R}^m \to \mathbb{R}^m$ is orthogonal projection onto the subspace spanned by $(\epsilon_1, \ldots, \epsilon_r)$

Quasi-Inverse Via Singular Value Decomposition

- Let the singular value decomposition of $L : X \to Y$ be
$$L = W\Sigma V^*,$$
- For each $1 \le k \le m$, let $e_k = V(\epsilon_k)$
- For each $1 \le j \le n$, let $f_j = W(\epsilon_j)$
- Then for any $x = e_1x^1 + \cdots + e_mx^m \in X$,
$$L(x) = L(e_1x^1 + \cdots + e_mx^m) = f_1s_1x^1 + \cdots + f_rs_rx^r$$
- Therefore, for any $y = f_1y^1 + \cdots + f_ny^n \in Y$,
$$L^+(y) = L^+(f_1y^1 + \cdots + f_ny^n) = e_1s_1^{-1}y^1 + \cdots + e_rs_r^{-1}y^r$$
- In other words,
$$L^+ = W\Sigma^+V^*$$

Image of Unit Ball

- The closed unit ball centered at the origin in $\mathbb{R}^n$ is

$$B = \{x \in \mathbb{R}^n : \; x \cdot x \leq 1\}$$

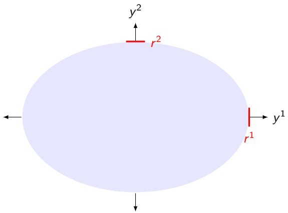
- Consider the image of B under a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$
- If A is diagonal, then

$$y = \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{bmatrix} = A \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} = \begin{bmatrix} r^1 & 0 & \cdots & 0 \\ 0 & r^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r^n \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} = \begin{bmatrix} r^1 x^1 \\ r^2 x^2 \\ \vdots \\ r^n x^n \end{bmatrix}$$

- Therefore, $y \in AB$ if and only if

$$1 \geq (x^1)^2 + \cdots + (x^n)^2 = \left(\frac{y^1}{r^1}\right)^2 + \cdots + \left(\frac{y^n}{r^n}\right)^2$$

Ellipse



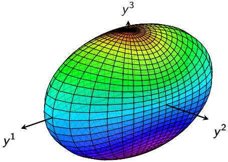
- If

$$y = \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} = \begin{bmatrix} r^1 & 0 \\ 0 & r^2 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = Ax$$

  then

$$x \in B \iff \frac{(y^1)^2}{(r^1)^2} + \frac{(y^2)^2}{(r^2)^2} \leq 1$$

3-Dimensional Ellipsoid



$$\frac{(y^1)^2}{(r^1)^2} + \frac{(y^2)^2}{(r^2)^2} + \frac{(y^3)^2}{(r^3)^2} \leq 1$$

n-Dimensional Ellipsoid in $\mathbb{R}^n$

- Given $r^1, \dots, r^n \neq 0$,

$$E = \left\{ (y^1, \dots, y^n) \in \mathbb{R}^n : \ \frac{(y^1)^2}{(r^1)^2} + \cdots + \frac{(y^n)^2}{(r^n)^2} \leq 1 \right\}$$

  is called an n-dimensional **ellipsoid**

- If A is a diagonal matrix with nonzero diagonal entries $r^1, \dots, r^n$, then

$$\begin{aligned} AB &= E \\ &= \{ y \in \mathbb{R}^n : \ (A^{-1}y, A^{-1}y) \leq 1 \} \end{aligned}$$

Ellipsoids in Inner Product Space

- A subset E of an n-dimensional real inner product space is an n-dimensional **ellipsoid** if there is a unitary basis $(u_1, \ldots, u_n)$ and nonzero scalars $r^1, \ldots, r^n$ such that

$$E = \left\{ y^1 u_1 + \cdots + y^n u_n : \; \frac{(y^1)^2}{(r^1)^2} + \cdots + \frac{(y^n)^2}{(r^n)^2} \leq 1 \right\}$$

- A subset E of an n-dimensional realinner product space is an k-dimensional **ellipsoid** if there is a unitary set $(u_1, \ldots, u_k)$ and nonzero scalars $r^1, \ldots, r^k$ such that

$$E = \left\{ y^1 u_1 + \cdots + y^n u_k : \; \frac{(y^1)^2}{(r^1)^2} + \cdots + \frac{(y^k)^2}{(r^k)^2} \leq 1 \right\}$$

Unitary Transformation of Ball is Ball

- If X and Y are inner product spaces with the same dimension, a map $U : X \to Y$ is a unitary transformation, if, for any $v \in X$,

$$(U(x), U(x))_Y = (x, x)_X$$

- Therefore, if

$$B_X = \{x \in X : \ (x, x) = 1\},$$

then

$$U(B_X) \subset B_Y$$

- On the other hand, if $y \in B_Y$, then $U^*(y)) \in B_X$ and $U(U^*(x)) = x$, which implies

$$B_Y \subset U(B_X)$$

- It follows that $U(B_X) = B_Y$

Singular Value Decomposition

- Let X and Y be real inner product spaces such that $\dim(X) = m$ and $\dim(Y) = n$
- $L : X \to Y$ be a linear transformation
- The singular value decomposition of L can be described as follows:
  - There exists a unitary basis $(e_1, \ldots, e_m)$ of X and a unitary basis $(f_1, \ldots, f_n)$ of Y such that if $r = \text{rank}(L)$, then

    $$L(e_k) = \begin{cases} s_k f_k & \text{if } 1 \leq k \leq r \\ 0 & \text{if } r+1 \leq k \leq m \end{cases},$$

    where $s_1, \ldots, s_n$ are the singular values of L
  - In particular, $(e_1, \ldots, e_r)$ is a unitary basis of $(\ker(L))^\perp$ and $(f_1, \ldots, f_r)$ is a unitary basis of $\text{image}(L)$

Linear Transformation of Ball is an Ellipsoid (Part 1)

- The unit ball is

$$B = \{x^1 e_1 + \cdots + x^n e_n : \ (x^1)^2 + \cdots + (x^n)^2 \leq 1\}$$

- If $x \in B$, then

$$\begin{aligned} L(x) &= x^1 L(e_1) + \cdots + x^n L(e_n) \\ &= s_1 x^1 f_1 + \cdots + s_r x^r f_r \\ &= y^1 f_1 + \cdots + y^r f_r, \end{aligned}$$

  where

$$\frac{(y^1)^2}{(s_1)^2} + \cdots + \frac{(y^r)^2}{(s_r)^2} = (x^1)^2 + \cdots + (x^r)^2 \leq 1$$

Linear Transformation of Ball is an Ellipsoid (Part 2)

- The set

$$E = \left\{ y^1 f_1 + \cdots + y^r f_r : \frac{(y^1)^2}{(s_1)^2} + \cdots + \frac{(y^n)^2}{(s_r)^2} = (x^1)^2 + \cdots + (x^r)^2 \leq 1 \right\} \subset \text{image}(L)$$

is an r-dimensional ellipsoid in Y such that

$$L(B_X) \subset E$$

Linear Transformation of Ball is an Ellipsoid (Part 3)

- Conversely, if $y = y^1 f_1 + \cdots + y^r f_r \in E$, then
$$L(x) = y,$$
where
$$x = \left(\frac{y^1}{s_1}\right) e_1 + \cdots + \left(\frac{y^r}{s_r}\right) e_n \in B$$
- It follows that $E \subset L(B)$
- Therefore, $E = L(B)$

Operator Norm of Linear Map

- Let X and Y be inner product spaces and $L : X \to Y$ be a linear map
- The **operator norm** of L is defined to be

$$\|L\| = \sup\{|L(x)| \ : \ x \in B_X\}$$

- Let $s_1 \leq s_2 \leq \cdots \leq s_r$ be the singular values of L
- For any $x = x^1 e_1 + \cdots + x^m e_m \in B$,

$$\begin{aligned}(L(x), L(x)) &= (x^1 s_1 f_1 + \cdots + x^r s_r f_r, x^1 s_1 f_1 + \cdots + x^r s_r f_r) \\ &= (s_1)^2 (x^1)^2 + \cdots + (s_r)^2 (x^r)^2 \\ &\leq (s_r)^2 ((x^1)^2 + \cdots + (x^r)^2) \\ &\leq (s_r)^2\end{aligned}$$

- Moreover,

$$(L(e_r), L(e_r)) = (s_r f_r, s_r f_r) = (s_r)^2$$

- Therefore, $\|L\|$ is equal to the largest singular value of L

Change of Basis Formula

- Let $L : X \to X$ be a linear endomorphism (codomain is domain)
- Given a basis $E(e_1, \dots, e_m)$ of X, there is a matrix M such that

$$L(e_k) = M^j_k e_j, i.e., L(E) = EM$$

- If $F = (f_1, \dots, f_m)$ is another basis such that

$$f_k = A^j_k e_j, \; i.e., F = EA,$$

then

$$L(F) = L(EA) = L(E)A = EMA = FA^{-1}MA$$

Trace of a Linear Endomorphism

- If $L(E) = EM$, then the trace of L is defined to be
$$\text{trace}(L) = M^1_1 + \cdots + M^m_m$$
- If $L(F) = EN$, then $N = A^{-1}MA$, i.e.,
$$N^l_k = (A^{-1})^l_i M^i_j A^j_k$$
- Therefore,
$$\begin{aligned} N^1_1 + \cdots + N^m_m &= N^k_k \\ &= (A^{-1})^k_i M^i_j A^j_k \\ &= A^j_k (A^{-1})^k_i M^i_j \\ &= \delta^j_i M^i_j \\ &= M^j_j \\ &= M^1_1 + \cdots + M^m_m \end{aligned}$$
- The definition of trace(L) does not depend on the basis used

Frobenius Norm of a Linear Transformation

- Let X and Y be real inner product spaces
- Let $L : X \to Y$ be a linear map
- Recall that the adjoint of L is the map $L^* : Y \to X$ such that for any $x \in X$ and $y \in Y$,

$$(L(x), y) = (x, L^*(y))$$

- The **Frobenius norm** or **Hilbert-Schmidt norm** of L is defined to be $\|L\|_2$, where

$$\|L\|_2^2 = \text{trace}(L^* L)$$

Frobenius Norm With Respect to Basis

- Let $(e_1, \ldots, e_m)$ be a unitary basis of X and $(f_1, \ldots, f_n)$ be a unitary basis of Y such that

$$L(e_k) = \begin{cases} s_k f_k & \text{if } 1 \le k \le r \\ 0 & \text{if } r+1 \le k \le m, \end{cases}$$

- The adjoint of L is given by

$$L^*(f_k) = \begin{cases} s_k e_k & \text{if } 1 \le k \le r \\ 0 & \text{if } r+1 \le k \le n \end{cases}$$

- Therefore,

$$L^*L(e_k) = \begin{cases} s_k^2 e_k & \text{if } 1 \le k \le r \\ 0 & \text{if } r+1 \le k \le m \end{cases}$$

- It follows that

$$\|L\|_2^2 = \text{trace}(L^*L) = s_1^2 + \cdots + s_r^2$$

- Observe that the operator norm is always less than or equal to the Frobenius norm,

Solving a Linear System with Errors

- Let $L : X \to Y$ be a linear map between inner product spaces
- Suppose that, given $y \in Y$, we want to solve

$$L(x) = y,$$

for x but the exact value of y is not known
- If the measured value of y is $y + \Delta y$ and

$$x + \Delta x = L^{-1}(y + \Delta y),$$

then

$$\Delta x = L^{-1}(\Delta y)$$

- The relative error of x can ye estimated in terms of the relative error of y:

$$\frac{|\Delta x|}{|x|} = \frac{|L^{-1}(\Delta y)|}{|y|}\frac{|y|}{|x|} = \frac{|L^{-1}(\Delta y)|}{|y|}\frac{|L(x)|}{|x|} \leq \|L^{-1}\|\|L\|\frac{|\Delta y|}{|y|}$$

Condition Number of Linear Map

- $\|L^{-1}\|\|L\|$ is the **condition number** of the linear map
- It shows how sensitive the error in x is to the error in y
- A linear map is **ill-conditioned** if the condition number is large
- The condition number can be changed by changing the inner product