MA-GY 7043: Linear Algebra II

Dual Vector Space
Tensors
Bilinear Tensors and Quadratic Forms
Complex Tensors

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Outline I

Dual Vector Space

- Let V be an n-dimensional real vector space
- The dual vector space is the vector space V^* of all linear functions on V
- If $\ell \in V^*$, then it is a function
$$\ell : V \to \mathbb{F}$$
such that if $a, b \in \mathbb{F}$ and $v, w \in V$, then
$$\ell(av + bw) = a\ell(v) + b\ell(w)$$
- For each $\ell^1, \ell^2 \in V^*$ and $a_1, a_2 \in \mathbb{F}$, the function $a_1\ell^1 + a_2\ell^2$ is also linear and therefore an element of V^*
- Therefore, V^* is a vector space
- For convenience, we will denote the value of ℓ with input v by any of the following:
$$\langle \ell, v \rangle = \langle v, \ell \rangle = \ell(v)$$
- An element of V^* can be called a **dual vector**, **covector**, or 1-**tensor**

Covector with respect to a Basis

- Let $(e_1, \ldots, e_m)$ be a basis of V
- Any $\ell \in V$ is uniquely determined by its values for the basis elements
- If
$$\ell(e_1) = b_1, \ldots, \ell(e_m) = b_m,$$
then for any $v = e_1 a^1 + \cdots + a_m a^m$,
$$\begin{aligned} \ell(v) &= \ell(e_1 a^1 + \cdots + a_m a^m) \\ &= \ell(e_1) a^1 + \cdots + \ell(e_m) a^m \\ &= b_1 a^1 + \cdots + b_m a^m \end{aligned}$$

Dual Basis

- Let $(e_1, \ldots, e_m)$ be a basis of V
- For each $1 \leq j \leq m$, there is an element $\epsilon^i \in V^*$ given by

$$\langle e_j, \epsilon^i \rangle = \delta^i_j$$

- In other words, if $v = e_1 a^1 + \cdots + e_m a^m$, then

$$\begin{aligned}
\langle v, \epsilon^i \rangle &= \langle e_1 a^1 + \cdots + e_m a^m, \epsilon^i \rangle \\
&= \langle e_1, \epsilon^i \rangle a^1 + \cdots + \langle e_m, \epsilon^m \rangle a^m \\
&= a^i
\end{aligned}$$

- In particular, for any $v \in V$,

$$v = e_1 \langle \epsilon^1, v \rangle + \cdots + e_m \langle \epsilon^m, v \rangle$$

- $(\epsilon^1, \ldots, \epsilon^m)$ is called the **dual basis** of the basis $(e_1, \ldots, e_m)$

Dual Basis is Basis of Dual Vector Space

- Given a basis $(e_1, \ldots, e_m)$ of V and its dual basis $(\epsilon^1, \ldots, \epsilon^m)$, there is a linear map

$$
\begin{aligned}
V^* &\to \mathbb{F}^n \\
\ell &\mapsto \langle \ell, e_1 \rangle \epsilon^1 + \cdots + \langle \ell, e_m \rangle \epsilon^m
\end{aligned}
$$

- Conversely, there is a linear map

$$
\begin{aligned}
\mathbb{F}^n &\to V^* \\
(b_1, \ldots, b_n) &\mapsto b_1 \epsilon^1 + \cdots + b_m \epsilon^m
\end{aligned}
$$

- These two maps are inverses of each other
- Therefore, the maps are isomorphhism
- It follows that

$$\dim(V^*) = \dim(V)$$

and $(\epsilon^1, \ldots, \epsilon^m)$ is a basis of V^*

Dual of Dual Vector Space

- The dual of V^* is the space of all linear functions $\nu : V^* \to \mathbb{F}$
- There is a natural (basis-independent) map

$$F : V \to V^{**},$$

where for each $v \in V$, $F(v) : V^* \to \mathbb{F}$ is given by

$$\langle F(v), \ell \rangle = \langle v, \ell \rangle$$

- If $F(v) = 0$, then for any $\ell \in V^*$,

$$\langle v, \ell \rangle = \langle F(v), \ell \rangle = 0$$

and therefore $v = 0$
- It follows that F is a basis-independent isomorphism
- We will denote $F(v)$ by simply v

Dual or Transpose of a Linear Map

- Let $L : X \to Y$ be a linear map
- There is a naturally defined dual linear map

$$L^* : Y^* \to X^*,$$

where for any $\eta \in Y^*$,

$$L^*(\eta) = \eta \circ L$$

- In other words, for any $\eta \in Y^*$, $L^*(\eta) \in X^*$ is the function where for any $x \in X$,

$$\langle L^*(\eta), x \rangle = \langle \eta, L(x) \rangle$$

- L^* is called the **dual** or **transpose** of L
- If $L : X \to Y$ and $M : Y \to Z$ are linear maps, then

$$(M \circ L)^* = L^* \circ M^*$$

- $(L^*)^* = L$

Tensors

- A 0-tensor is a scalar
- Given $k > 0$, a k-tensor on a vector space V is a k-linear function on V,
$$\theta : V \times \cdots \times V \to \mathbb{F}$$
- For each $1 \le j \le k$, vectors $v_1, \ldots, v_k, w_j \in V$, and scalars $a^j, b^j \in \mathbb{F}$,
$$\begin{aligned} &\theta(v_1, \ldots, v_{j-1}, , a^j v_j + b^j w_j, v_{j+1}, \ldots, v_k) \\ &= a^j \theta(v_1, \ldots, v_{j-1}, , v_j, v_{j+1}, \ldots, v_k) \\ &\quad + b^j \theta(v_1, \ldots, v_{j-1}, , w_j, v_{j+1}, \ldots, v_k) \end{aligned}$$
- The space of all k-tensors on V is denoted $\bigotimes^k V^*$
- Examples
  - An inner product is a 2-tensor
  - An element of $\Lambda^m V^*$ (where $\dim(V) = m$) is an m-tensor

Symmetric and Antisymmetric Tensors

- A k-tensor θ is **symmetric** if for any permutation $\sigma \in S_k$,

$$\theta(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \theta(v_1, \ldots, v_k)$$

- The space of all symmetric k-tensors is denoted $S^k V^*$
- A k-tensor θ is **antisymmetric** if for any permutation $\sigma \in S_k$,

$$\theta(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \epsilon(\sigma)\theta(v_1, \ldots, v_k)$$

- The space of all antisymmetric k-tensors is denoted $\Lambda^k V^*$
- An inner product is a symmetric 2-tensor
- An element of $\Lambda^m V^*$ is an antisymmetric m-tensor

2-Tensor With Respect to Basis

- Let $B : V \times V \to \mathbb{R}$ be a symmetric 2-tensor
- Let $(e_1, \ldots, e_m)$ be a basis of V
- For each $1 \leq i, j \leq m$, let

$$M_{ij} = B(e_i, e_j)$$

- If $v = e_j a^j$ and $w = e_k b^k$, then

$$\begin{aligned} B(v, w) &= B(e_j a^j, e_k b^k) \\ &= a^j b^k M_{jk} \end{aligned}$$

- Therefore, with respect to a basis of V, the bilinear form B is uniquely determined by the matrix M and

$$B(Ea, Eb) = a^T M b$$

- If B is symmetric, then

$$M_{ij} = B(e_i, e_j) = B(e_j, e_i) = M_{ji}$$

Quadratic Form on Real Vector Space

- A function $Q : V \to \mathbb{R}$ is a **quadratic form** if with respect to a basis $(e_1, \ldots, e_m)$,

$$Q(e_j a^j) = P(a^1, \ldots, a^n),$$

where P is a homogeneous quadratic polynomial, i.e, every term of P has degree 2

- In particular, there exists a symmetric matrix M such that

$$Q(Ea) = Q(e_j a^j) = M_{jk} a^j a^k = a^T M a = B(Ea, Eb),$$

where

$$M_{jk} = \frac{1}{2} \frac{\partial^2 Q}{\partial a^j \partial a^k}$$

- If B is a symmetric 2-tensor, the function

$$Q(v) = B(v, v)$$

is a quadratic form

Equivalence of Quadratic Forms and Symmetric 2-Tensors

- Let $Q : V \to \mathbb{R}$ be a quadratic form such that with respect to a basis $(e_1, \ldots, e_m)$ of V,

$$Q(e_1 a^1 + \cdots + e_m a^m) = a^j a^k M_{jk},$$

  for a symmetric matrix M
- Define $B \in S^2 V^*$ by setting

$$B(e_j, e_k) = M_{jk}$$

- Then if $v = e_1 a^1 + \cdots + e_m a^m$, then

$$\begin{aligned} B(v, v) &= B(e_j a^j, e_k a^k) \\ &= a^j a^k B(e_j, e_k) \\ &= a^j a^k M_{jk} \\ &= Q(v) \end{aligned}$$

Signature of a Symmetric Matrix

- Recall that if M is a real symmetric matrix, there exists an orthogonal matrix U such that

$$D = U^T M U$$

  is diagonal and the diagonal entries of D are the eigenvalues of M
- The signature of M is defined to be (p, q, r), where p is the number of positive eigenvalues, q is the number of negative eigenvalues, and r is the number of zero eigenvalues
- Since $p + q + r = \dim(V)$, it suffices to specify only (p, q)

Normal Form of a Symmetric Matrix

- Let $d_1, \ldots, d_m$ be the eigenvalues of M
- Let E be the matrix whose diagonal entries are

$$e_k = \begin{cases} |d_k|^{-1/2} & \text{if } d_k \neq 0 \\ 0 & \text{if } d_k = 0 \end{cases}$$

  and

$$V = UE$$

- Then

$$V^T MV = E^T U^T MUE = EDE = H,$$

  where H is a diagonal matrix, where

$$H_{kk} \begin{cases} 1 & \text{if } d_k > 0 \\ -1 & \text{if } d_k < 0 \\ 0 & \text{if } d_k = 0 \end{cases}$$

Tensors on Complex Vector Space

- Let V be a complex vector space
- A $(1,0)$-tensor is a linear function $\ell : V \to \mathbb{C}$, i.e.,

$$\ell(av + bw) = a\ell(v) + b\ell(w)$$

- Let $V^{(1,0)}$ denote the space of all $(1,0)$-tensors on V
- A $(0,1)$-tensor is a conjugate linear function $\ell : V \to \mathbb{C}$, i.e.,

$$\ell(av + bw) = \bar{a}\ell(v) + \bar{b}\ell(w)$$

- Let $V^{(0,1)}$ denote the space of all $(0,1)$-tensors on V
- Both $V^{(1,0)}$ and $V^{(0,1)}$ are complex vector spaces
- If ℓ is a $(1,0)$-tensor, then $\bar{\ell}$ is a $(0,1)$-tensor and therefore the map

$$V^{(1,0)} \to V^{(0,1)}$$
$$\ell \mapsto \bar{\ell}$$

is a conjugate linear isomorphism

(k, l)-tensors on Complex Vector Space

- Given nonnegative integers k, l such that $k + l \leq \dim(V)$, a function

$$V \times \cdots \times V \to \mathbb{C}$$
$$(v_1, \ldots, v_k, v_{k+1}, \cdots, v_{k+l}) rightarrow T(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l})$$

is a (k, l)-tensor if for each $1 \leq i \leq k$, the function

$$v_i \mapsto T(v_1, \ldots, v_{i-1}, v_i, \ldots, v_k, \ldots, v_{k+l})$$

is linear and for each $k + 1 \leq i \leq l$, the function

$$v_i \mapsto T(v_1, \ldots, v_k, \ldots, v_{i-1}, v_i, \ldots, v_{k+l})$$

is conjugate linear