MA-GY 7043: Linear Algebra II

Isometries of Euclidean Space

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Outline I

Isometries

- Let V be a real inner product space
- Given $v, w \in V$, the distance between v and w is defined to be

$$d(v, w) = |v - w| = \sqrt{(v - w, v - w)}$$

- A map $F : V \to V$ is called an *isometry* if for any $v, w \in V$,

$$d(F(v), F(w)) = d(v, w)$$

  or, equivalently,

$$|F(v) - F(w)| = |v - w|$$

Examples of Isometries

- Given $u \in V$, define **translation by** u to be the map

$$T_u(v) = v + t$$

- Given an orthogonal map R with $\det(R) = 1$, let

$$F(v) = R(v)$$

  Such a map is called a **rotation**
- Given any $u \in V$ and orthogonal map R, let

$$F(v) = T_u \circ R(v) = T_u(Rv) = u + R(v)$$

  Any such map is called a **rigid motion**
- For any $v, w \in V$ and rigid motion F,

$$\begin{aligned} |F(v) - F(w)| &= |(u + R(v)) - (u + R(w))| \\ &= |R(v - w)| \\ &= |v - w| \end{aligned}$$

  and therefore a rigid motion is an isometry

Group of Rigid Motions

- Composition of rigid motions is a rigid motion:
  - If
  $$F_1(v) = u_1 + R_1(v) \text{ and } F_1(v) = u_1 + R_1(v),$$
  then
  $$\begin{aligned} F_2 \circ F_1(v) &= u_2 + R_2(F_1(v)) \\ &= u_2 + R_2(u_1 + R_1(v)) \\ &= u_2 + R_2(u_1) + (R_2 \circ R_1)(v) \end{aligned}$$
  - Since the composition of rotations is a rotation, it follows that $F_2 \circ F_1$ is a rigid motion
  - The identity is a rigid motion
  - Recall that the inverse of a rotation is a rotation
  - If $F(v) = u + R(v)$ is a rigid motion, then it is an invertible map, where the inverse is given by
  $$F^{-1}(w) = R^{-1}(w - u) = -R^{-1}(u) + R^{-1}(w)$$

An Isometry Preserves the Inner Product (Part 1)

- Let $F : V \to V$ be an isometry, i.e, for any $v, w \in V$,

$$|F(v) - F(w)| = |v - w|$$

- The map $G : V \to V$ given by

$$G(v) = F(v) - F(0)$$

  is an isometry that satisfies $G(0) = 0$

An Isometry Preserves the Inner Product (Part 2)

- Recall that for any $v, w \in V$,

$$\begin{aligned}(v, w) &= \frac{1}{2}((v, v) + (w, w) - (v - w, v - w)) \\ &= \frac{1}{2}(|v|^2 + |w|^2 - |v - w|^2)\end{aligned}$$

- Therefore, for any $v, w \in V$,

$$\begin{aligned}(G(v), G(w)) &= \frac{1}{2}(|G(v) - G(0)|^2 \\ &\quad + |G(w) - G(0)|^2 - |G(v) - G(w)|^2) \\ &= \frac{1}{2}(|v - 0|^2 + |w - 0|^2 - |v - w|^2) \\ &= (v, w)\end{aligned}$$

An Isometry is a Rigid Motion

- Let $(e_1, \ldots, e_n)$ be an orthonormal basis of V
- Let $f_1 = G(e_1), \ldots, f_n = G(e_n)$
- For any $1 \leq i, j \leq n$,

$$(f_i, f_j) = (G(e_i), G(e_j)) = (e_i, e_j) = \delta_{ij}$$

- Therefore, $(f_1, \ldots, f_n)$ is an orthonormal basis of V
- For each $v = e_1 a^1 + \cdots + e_n a^n \in V$ and $1 \leq j \leq n$,

$$(f_j, G(v)) = (G(e_j), G(v)) = (e_j, v) = a^j$$

  and therefore

$$G(e_1 a^1 + \cdots + e_n a^n) = f_1 a^1 + \cdots + f_n a^n,$$

- It follows that G is a linear isometry and therefore a rotation R
- This implies that

$$F(v) = u + R(v),$$

  where $u = F(0)$

Matrix Representation of Rigid Motions

- A rigid motion $F : \mathbb{R}^n \to \mathbb{R}^n$ is not necessarily linear and therefore cannot be written as an n-by-n matrix
- It can, however, be written as an $(n+1)$-by-$(n+1)$ matrix

Euclidean Space

- Define n-dimensional Euclidean space to be

$$\mathbb{E} = \{(1, x^1, \ldots, x^n) \in \mathbb{R}^{n+1}\}$$

- $\mathbb{E}$ is parallel to the inner product space

$$\mathbb{V} = \{(0, x^1, \ldots, x^n) \in \mathbb{R}^{n+1}\}$$

- Given any two points $p, q \in \mathbb{E}$, the vector from p to q is

$$v = q - p \in \mathbb{V}$$

and the distance between p and q is

$$d(p, q) = |q - p|$$

Isometries of Euclidean Space

- Consider $(n+1)$-by-$(n+1)$ matrices of the form

$$M = \left[\begin{array}{c|c} 1 & 0 \\ \hline u & R \end{array}\right] = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline u^1 & R^1_1 & \cdots & R^1_n \\ \vdots & \vdots & & \vdots \\ u^n & R^n_1 & \cdots & R^n_n \end{array}\right],$$

  where $u \in \mathbb{R}^n$ and $R \in O(n)$
- Given $x \in \mathbb{R}^n$, observe that

$$M \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ u^1 & R^1_1 & \cdots & R^1_n \\ \vdots & \vdots & & \vdots \\ u^n & R^n_1 & \cdots & R^n_n \end{bmatrix} \begin{bmatrix} 1 \\ x^1 \\ \vdots \\ x^n \end{bmatrix} = \begin{bmatrix} 1 \\ u + Rx \end{bmatrix}$$

- The set of all such matrices forms a group that is isomorphic to the group of rigid motions

Affine Transformations

- Rigid motions are special cases of affine transformations
- A map $F : \mathbb{R}^n \to \mathbb{R}^n$ is an **affine transformation** if there exists $u \in \mathbb{R}^n$ and a linear isomorphism $L : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$F(v) = u + L(v)$$

- Geometric idea: Move the origin and apply a linear transformation