MA-GY 7043: Linear Algebra II

Affine Geometry

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April 24, 2025

Outline I

Geometry of Affine n-Space

- Affine n-space is an n-dimensional flat space of points
- Given any two points, there is a unique line segment connecting them
- If a point p is designated as the start and another point q as the end, then there is an arrow v connecting them
- Given a point p and an arrow v, there is a unique point q such that if the start of the arrow is at p, then the end of the arrow is q
- Arrows can be added and rescaled, i.e., they are vectors

Abstract Definition of Affine Space

- Associated with an affine n-space $\mathbb{A}$ is an n-dimensional vector space $\mathbb{V}$, where:
  - Given any $p, q \in \mathbb{A}$, there is a unique vector v from p to q, denoted
  $$v = q - p$$
  - Given any $p \in \mathbb{A}$ and $v \in \mathbb{V}$, there is a unique point q, denoted
  $$q = p + v$$
  such that
  $$v = q - p$$
- Assumptions:
$$p + 0 = p$$
$$p - p = 0$$
$$p + (v + w) = (p + v) + w$$
- $\mathbb{V}$ is called the **tangent space** of $\mathbb{A}$

Affine Maps

- Let $\mathbb{A}$ and $\mathbb{B}$ be affine spaces with tangent spaces $\mathbb{V}$ and $\mathbb{W}$, respectively
- A map $F : \mathbb{A} \to \mathbb{B}$ is **affine** if there exists a linear map

$$dF : \mathbb{V} \to \mathbb{W}$$

  such that for each $p \in \mathbb{A}$ and $v \in \mathbb{V}$,

$$F(p + v) = F(p) + dF(v)$$

  or, equivalently, for any $p, q \in \mathbb{A}$,

$$F(q) = F(p) + dF(q - p)$$

- dF is called the **differential** of F

Explicit Definition of Affine n-Space

- Let X be an $(n+1)$-dimensional vector space
- Let $\xi \in X^* \backslash \{0\}$ and

$$\mathbb{A} = \{x \in X : \langle \xi, x \rangle = 1\}$$
$$\mathbb{V} = \{v \in X : \langle \xi, v \rangle = 0\}$$

- $\mathbb{V}$ is an n-dimensional linear subspace of X
- $\mathbb{A}$ is a hyperplane parallel to and not equal to $\mathbb{V}$
- Affine arithmetic
  - Addition of vector to point

$$\mathbb{A} \times \mathbb{V} \to \mathbb{A}$$
$$(x, v) \mapsto x + v$$

  - Subtraction of points

$$\mathbb{A} \times \mathbb{A} \to \mathbb{V}$$
$$(x_1, x_2) \mapsto x_2 - x_1$$

Affine Maps

- Let X and Y be vector spaces
- Given $\xi \in X^*\backslash\{0\}$ and $\eta \in Y^*\backslash\{0\}$, let

$$\mathbb{A} = \{x \in X : \ \langle \xi, x \rangle = 1\} \text{ and } \mathbb{V} = \{x \in X : \ \langle \xi, x \rangle = 0\}$$
$$\mathbb{B} = \{y \in Y : \ \langle \eta, y \rangle = 1\} \text{ and } \mathbb{W} = \{y \in Y : \ \langle \eta, y \rangle = 0\}$$

- A map $F : \mathbb{A} \to \mathbb{B}$ is **affine** if there exists a linear map

$$\widehat{F} : X \to Y$$

such that

$$\widehat{F}(\mathbb{A}) \subset \mathbb{B}$$
$$F = \widehat{F}\Big|_{\mathbb{A}}$$

- The differential of F is

$$dF = \widehat{F}\Big|_{\mathbb{V}}$$

Composition of Affine Maps is an Affine Map

- If
$$F_1 : \mathbb{A} \to \mathbb{B} \text{ and } F_2 : B \to \mathbb{C}$$
are affine maps, then so is
$$F_2 \circ F_1 : \mathbb{A} \to \mathbb{C},$$
because if
$$\widehat{F}_1 : X \to Y \text{ and } \widehat{F}_2 : Y \to Z,$$
are linear, then so is
$$\widehat{F}_2 \circ \widehat{F}_1 : X \to Z$$

F is Affine $\iff$ $\widehat{F}^*\eta = \xi$

- For any $x \in \mathbb{V}$, $\widehat{F}(x) \in \mathbb{W}$ and therefore
$$\langle \widehat{F}^*\eta, x\rangle = \langle \eta, \widehat{F}(x)\rangle = 0$$
and therefore, there exists $c \in \mathbb{F}$ such that
$$F^*\eta = c\xi$$
- For any $x \in \mathbb{A}$, $\widehat{F}(x) \in \mathbb{B}$ and therefore
$$1 = \langle \eta, \widehat{F}(x)\rangle = \langle \widehat{F}^*\eta, x\rangle = \langle c\xi, x\rangle = c$$
- It follows that $\widehat{F}^*\eta = \xi$
- Conversely, if $\widehat{F}^*\eta = \xi$, then $F(\mathbb{A}) \subset \mathbb{B}$ and $F = \widehat{F}\Big|_{\mathbb{A}}$ is an affine map

Affine Transformations

- An affine map $F : \mathbb{A} \to \mathbb{A}$ is invertible if and only if the linear map $\widehat{F} : X \to X$ is invertible
- Such a map is called an **affine transformation**
- The inverse of an affine transformation is an affine transformation
- Therefore, the set of all affine transformations is a group

Affine Space With Respect to Basis

- Let $(e_0, \ldots, e_m)$ be a basis of X and $(\epsilon^0, \ldots, \epsilon^m)$ be the dual basis such that $\xi = \epsilon^0$
- Then

$$\begin{aligned} \mathbb{A} &= \{x \in X : \ \langle \epsilon^0, x \rangle = 1\} \\ &= \{e_0 + e_1 a^1 + \cdots + e_m a^m : \ (a^1, \ldots, a^m) \in \mathbb{F}^m\} \\ \mathbb{V} &= \{v \in X : \ \langle \epsilon^0, v \rangle = 0 \\ &= \{e_1 a^1 + \cdots + e_m a^m : \ (a^1, \ldots, a^m) \in \mathbb{F}^m\} \end{aligned}$$

- Let $(f_0, \ldots, f_n)$ be a basis of Y and $(\phi^0, \ldots, \phi^n)$ be the dual basis such that $\eta = \phi^0$
- Then

$$\begin{aligned} \mathbb{B} &= \{y \in Y : \ \langle \phi^0, y \rangle = 1\} \\ &= \{f_0 + f_1 a^1 + \cdots + f_n a^n : \ (a^1, \ldots, a^n) \in \mathbb{F}^n\} \\ \mathbb{W} &= \{w \in Y : \ \langle \phi^0, w \rangle = 0\} \\ &= \{f_1 a^1 + \cdots + f_n a^n : \ (a^1, \ldots, a^n) \in \mathbb{F}^n\} \end{aligned}$$

Matrix Representation of an Affine Transformation (Part 1)

- Let $F : \mathbb{A} \to \mathbb{B}$ be the affine map and $\widehat{F} : X \to Y$ be the corresponding linear map
- Let $\widehat{M}$ be the matrix for the linear map $\widehat{F}$ with respect to the basis $(e_0, \ldots, e_n)$ defined above
- Since $\widehat{F}(e_0) \in \mathbb{B}$,

$$1 = \langle \phi^0, \widehat{F}(e_0) \rangle = \langle \phi^0, f_0 \widehat{M}_0^0 + \cdots + f_n \widehat{M}_0^n \rangle = \widehat{M}_0^0$$

- Since for each $1 \leq k \leq m$, $F(e_k) \in \mathbb{W}$,

$$0 = \langle \phi^0, \widehat{F}(e_k) \rangle = \langle \phi^0, f_0 \widehat{M}_k^0 + \cdots + f_n \widehat{M}_0^n \rangle = \widehat{M}_k^0$$

Matrix Representation of an Affine Transformation (Part 2)

- It follows that for any $x = e_0 + e_1 a^1 + \cdots + e_m a^m \in \mathbb{A}$,

$$
\begin{aligned}
F(x) &= \widehat{F}(e_0 + e_1 a^1 + \cdots + e_m a^m) \\
&= f_0 + f_1 \widehat{M}_0^1 + \cdots f_n \widehat{M}_0^n + \sum_{k=1}^{n} f_k (\widehat{M}_1^k a^1 + \cdots + \widehat{M}_m^k a^m) \\
&= \begin{bmatrix} f_0 & f_1 & \cdots & f_n \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \widehat{M}_0^1 & \widehat{M}_1^1 & \cdots & \widehat{M}_m^1 \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{M}_0^n & \widehat{M}_1^n & \dots & \widehat{M}_m^n \end{bmatrix} \begin{bmatrix} 1 \\ a^1 \\ \vdots \\ a^m \end{bmatrix} \\
&= \begin{bmatrix} f_0 & f_1 & \cdots & f_n \end{bmatrix} \begin{bmatrix} 1 \\ \widehat{M}_0^1 + \widehat{M}_1^1 a^1 + \cdots + \widehat{M}_m^1 a^m \\ \vdots \\ \widehat{M}_0^n + \widehat{M}_1^n a^1 + \cdots + \widehat{M}_m^n a^m \end{bmatrix}
\end{aligned}
$$

Matrix Representation of an Affine Transformation (Part 2)

- It follows that for any $x = e_0 + e_1 a^1 + \cdots + e_m a^m \in \mathbb{A}$,

$$\begin{aligned} F(x) &= \widehat{F}(e_0 + e_1 a^1 + \cdots + e_m a^m) \\ &= F(e_0) + \widehat{F}(e_1)a^1 + \cdots + \widehat{F}(e_m)a^m \\ &= f_0 + f_1 \widehat{M}_0^1 + \cdots f_n \widehat{M}_0^n) + \sum_{k=1}^{n} f_k(\widehat{M}_1^k a^1 + \cdots + \widehat{M}_m^k a^m) \\ &= \begin{bmatrix} f_0 & f_1 & \cdots & f_n \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \widehat{M}_0^1 & \widehat{M}_1^1 & \cdots & \widehat{M}_m^1 \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{M}_0^n & \widehat{M}_1^n & \ldots & \widehat{M}_m^n \end{bmatrix} \begin{bmatrix} 1 \\ a^1 \\ \vdots \\ a^m \end{bmatrix} \end{aligned}$$

Matrix Representation of the Differential

- Recall that $dF = \widehat{F}\Big|_{\mathbb{V}}$
- Therefore, for any $v = e_1 \dot{a}^1 + \cdots + e_m \dot{a}^m \in \mathbb{V}$,

$$\begin{aligned} dF(v) &= \widehat{F}(e_1 \dot{a}^1 + \cdots + e_m \dot{a}^m) \\ &= \widehat{F}(e_1)\dot{a}^1 + \cdots + \widehat{F}(e_m)\dot{a}^m \\ &= \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} \begin{bmatrix} \widehat{M}_1^1 & \cdots & \widehat{M}_m^1 \\ \vdots & \vdots & \vdots \\ \widehat{M}_1^n & \dots & \widehat{M}_m^n \end{bmatrix} \begin{bmatrix} \dot{a}^1 \\ \vdots \\ \dot{a}^m \end{bmatrix} \end{aligned}$$

Affine Maps From $\mathbb{F}^m$ to $\mathbb{F}^n$

- Let $F : \mathbb{F}^m \to \mathbb{F}^n$ be an affine map
- Therefore, there exists an n-by-m matrix M and $v \in \mathbb{F}^n$ such that for each $x \in \mathbb{F}^m$,

$$F(x) = v + Mx.$$