MA-GY 7043: Linear Algebra II

Subspaces
Permutations
Alternating Multilinear functions

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Outline I

Subspace and its Dimension

- A subset T of a vector space X is a **subspace** of X if for any $p, q \in \mathbb{F}$ and $a, b \in T$,

$$pa + qb \in T$$

- If a subspace has at least one nonzero vector, then it is itself a vector space
- Define the dimension of a subspace S as follows:
  - If $S = \{\vec{0}\}$ then $\dim S = 0$
  - If $S \neq \{\vec{0}\}$, then S is a vector space and $\dim S$ is its dimension as a vector space

Kernel, Image, Rank of a Linear Map

- Associated to any linear map $P : Z \to Y$ are the following subspaces
  - The **kernel** of P is defined to be

$$\ker P = \{z \in Z \ : \ P(z) = \vec{0}\}$$

  - The **image** of P is defined to be

$$P(Z) = \{P(z) \ : \ z \in Z\} \subset Y$$

- The **rank** of P is

$$\operatorname{rank}(P) = \dim P(Z)$$

Example 1

- Define $Z : \mathbb{F}^2 \to \mathbb{F}^3$ to be

$$Z(x, y) = (x, y, 0), \text{ for all } (x, y) \in \mathbb{F}^2$$

- In other words,

$$Z\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- $\ker Z = \{0\}$
- $Z(\mathbb{F}^2) = \{(x, y, 0) \ : x, y, \in \mathbb{F}\} \subset \mathbb{F}^n$
  - A basis of $Z(\mathbb{F}^2)$ is $\{Z(e_1), Z(e_2)\} = \{(1, 0, 0), (0, 1, 0)\}$
- Therefore,

$$\dim \ker Z = 0$$

$$\operatorname{rank} Z = 2$$

Example 2

- Define $W : \mathbb{F}^2 \to \mathbb{F}^3$ to be
$$W(x, y) = (y, 0, 0), \text{ for all } (x, y) \in \mathbb{F}^2$$
- In other words,
$$W\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
- $\ker W = \{(x, 0) \ : \ x \in \mathbb{F}\}$
  - A basis of $\ker W$ is $\{(1, 0)\}$
- $W(\mathbb{F}^2) = \{(y, 0, 0) \ : \ y \in \mathbb{F}\}$
  - A basis of $W(\mathbb{F}^2)$ is $\{(1, 0, 0)\}$
- Therefore,
$$\dim \ker W = 1$$
$$\operatorname{rank} W = 1$$

Example 3

- Define $U : \mathbb{F}^2 \to \mathbb{F}^3$ to be

$$U(x, y) = (0, 0, 0), \text{ for all } (x, y) \in \mathbb{F}^2$$

- In other words,

$$U\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- $\ker U = \mathbb{F}^2$
- $U(\mathbb{F}^2) = \{(0, 0, 0\}$
- Therefore,

$$\dim \ker U = 2$$
$$\operatorname{rank} U = 0$$

Example 4

- Define $U : \mathbb{F}^3 \to \mathbb{F}^2$ to be
$$U(x, y, z) = (y, z), \text{ for all } (x, y, z) \in \mathbb{F}^3$$
- In other words,
$$U\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
- $\ker U = \{(x, 0, 0) \ : \ z \in \mathbb{F}\}$
  - A basis is $\{(1, 0, 0)\}$
- $U(\mathbb{F}^3) = \mathbb{F}^2$
- Therefore,
$$\dim \ker U = 1$$
$$\operatorname{rank} U = 2$$

Example 5

- Define $U : \mathbb{F}^3 \to \mathbb{F}^2$ to be

$$U(x, y, z) = (z, 0), \text{ for all } (x, y, z) \in \mathbb{F}^3$$

- In other words,

$$U\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- $\ker U = \{(x, y, 0) \ : \ x, y \in \mathbb{F}\}$
  - A basis is $\{(1, 0, 0), (0, 1, 0)\}$
- $U(\mathbb{F}^2) = \{(z, 0) \ : \ z \in \mathbb{F}\}$
  - A basis is $\{(1, 0)\}$
- Therefore,

$$\dim \ker U = 2$$
$$\operatorname{rank} U = 1$$

Example 6

- Define $U : \mathbb{F}^3 \to \mathbb{F}^2$ to be

$$T(x, y, z) = (0, 0, 0), \text{ for all } (x, y, z) \in \mathbb{F}^3$$

- In other words,

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- $\ker U = \mathbb{F}^3$
- $U(\mathbb{F}^3) = \{(0, 0, 0)\}$
- Therefore,

$$\dim \ker U = 3$$
$$\operatorname{rank} U = 0$$

Normal Form of a Linear Map

- Let $L : V \to W$ be a linear map
- Lemma: There exists a basis $(e_1, \ldots, e_m)$ of V and a basis $(f_1, \ldots, f_n)$ of W such that for each $1 \leq k \leq m$,

$$L(e_k) = \begin{cases} f_k & \text{if } 1 \leq k \leq r \\ 0_W & \text{if } r+1 \leq k \leq m \end{cases},$$

  where $r = \text{rank}(L)$
- In particular,

$$\ker(L) = \text{ span of } \{e_{r+1}, \ldots, e_m\} \text{ and } L(V) = \text{ span of } \{f_1, \ldots, f_r\}$$

- The matrix of L with respect to this basis is

$$M = \left[\begin{array}{c|c} I_{r\times r} & 0_{r\times m-r} \\ \hline 0_{n-r,r} & 0_{n-r,m-r} \end{array}\right]$$

Proof of Existence of Normal Form

- Let $s = \dim ker(L)$ and $r = \dim V - \dim \ker(L) = m - s$
- If $s > 0$, there exists a basis of $\ker(L)$, which will be denoted

$$(e_{m-s+1}, \ldots, e_m)$$

- This can be extended to a basis $(e_1, \ldots, e_r, e_{r+1}, \ldots, e_m)$ of V
- For each $1 \leq k \leq r$, let $f_k = L(e_k)$
- $(f_1, \ldots, f_r)$ is linearly independent
- It can be extended to a basis $(f_1, \ldots, f_n)$ of W
- It follows that

$$\begin{aligned} \dim \ker L + \operatorname{rank} L &= \dim \ker L + \dim L(V) \\ &= s + r = m \\ &= \dim V \end{aligned}$$

Corollary: Rank-Nullity Theorem

- Theorem: $\dim \ker(L) + \text{rank}(L) = \dim V$
- Proof: The normal form shows that if $\dim V = m$ and $\text{rank}(L) = r$, then $\dim \ker(L) = m - r$

Injective and Surjective Maps

- Consider a linear map $L : V \to W$
- $\dim \ker L = 0 \iff L$ is injective:

$$\begin{aligned} L(v_1) = L(v_2) &\iff L(v_2) - L(v_1) = 0_W \\ &\iff L(v_2 - v_1) = 0_W \\ &\iff v_2 - v_2 \in \ker L = \{0_V\} \\ &\iff v_2 = v_1 \end{aligned}$$

- $\operatorname{rank} L = \dim W \iff L$ is surjective:

$$\operatorname{rank} L = \dim W \iff \dim L(V) = \dim W \iff L(V) = W$$

Bijective Maps

- It also follows that

$$\begin{aligned} L : V \to W &\text{ is bijective} \\ \iff &\ \dim\ker(L) = 0 \text{ and } \operatorname{rank}(L) = \dim W \\ \iff &\ \dim V = \dim W \text{ and } \dim\ker L = 0 \\ \iff &\ \dim V = \dim W = \operatorname{rank} L \end{aligned}$$

Example (Part 1)

- Consider the map $L : \mathbb{F}^3 \to \mathbb{F}^2$ given by

$$L\left(\begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \begin{bmatrix} v^1 + 2v^2 + 3v^3 \\ 4v^3 \end{bmatrix}$$

- $\ker L = \{(v^1, v^2, v^3) \ : \ v^1 + 2v^2 = 0\}$
- A basis of $\ker L$ is $\{(-2, 1, 0)\}$
- A basis of $\mathbb{F}^3$ is $\{(0, 1, 0), (0, 0, 1), (-2, 1, 0)\}$
- A basis of $L(\mathbb{F}^3)$ is

$$\{L(0, 1, 0), L(0, 0, 1)\} = \{(2, 0), (3, 4)\}$$

Example (Part 2)

- If

$$\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \left[\begin{array}{c|c|c} 0 & 0 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \text{ and } \begin{bmatrix} f_1 & f_2 \end{bmatrix} = \left[\begin{array}{c|c} 2 & 3 \\ 0 & 4 \end{array}\right]$$

- Then

$$\begin{bmatrix} L(e_1) & L(e_2) & L(e_3) \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & 0 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

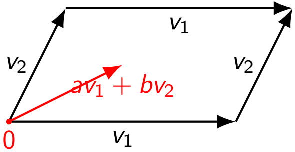
- And given any vector $v = e_1 a^1 + e_2 a^2 + e_3 a^3$,

$$L(v) = L(e_1)a^1 + L(e_2)a^2 + L(e_3)a^3 = f_1 a^2 + f_2 a^3 = FMa,$$

where

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

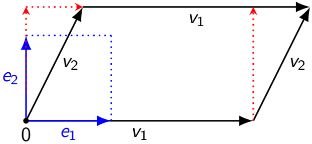
Parallelogram in Vector Space



- Let V be a 2-dimensional vector space
- Let $P(v_1, v_2)$ be the parallelogram with sides $v_1, v_2 \in V$.

$$P(v_1, v_2) = \{av_1 + bv_2 \ : \ 0 \leq a, b \leq 1\}.$$

Area of Parallelogram



- Let (e_1, e_2) be a basis of V
- Assume that the area of the parallelogram $P(e_1, e_2)$ is

$$A(e_1, e_2) = 1$$

- Let

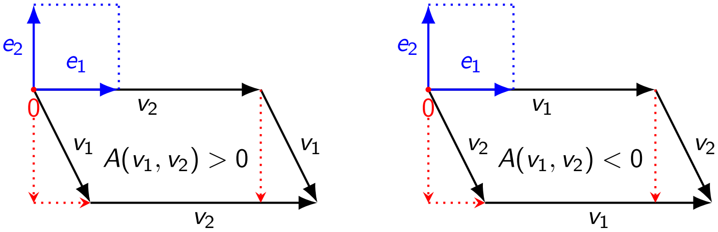
$$v_1 = we_1 \text{ and } v_2 = ae_1 + he_2$$

- With respect to this basis,
  - Height of $P(v_1, v_2)$ is h
  - Width of $P(v_1, v_2)$ is w
- The area of $P(v_1, v_2)$ is

$$A(v_1, v_2) = |h||w|$$

- The absolute values makes this formula hard to use

Oriented Area of Parallelogram

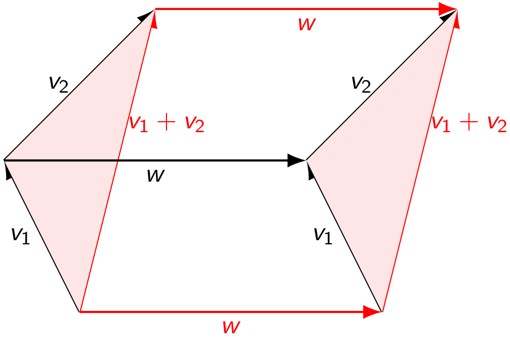


- Define oriented area of $P(v_1, v_2)$ to be

$$A(v_1, v_2) = \begin{cases} hw & \text{if } v_2 \text{ lies counterclockwise of } v_1 \\ -hw & \text{if } v_2 \text{ lies clockwise of } v_1 \end{cases}$$

- Oriented area, as a function of $v_1, v_2 \in V$ has nice properties

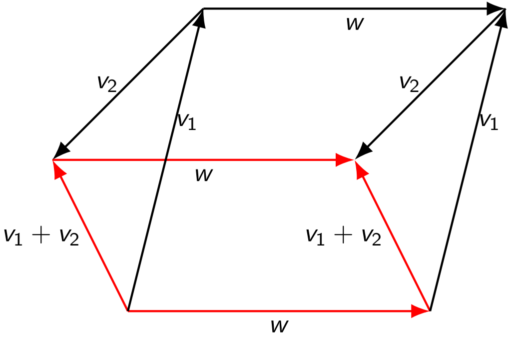
Oriented Area of Parallelograms with Parallel Bases



- If v_1 and v_2 both point upward relative to w, then

$$A(w, v_1 + v_2) = A(w, v_1) + A(w, v_2)$$

Oriented Area of Parallelograms with Parallel Bases



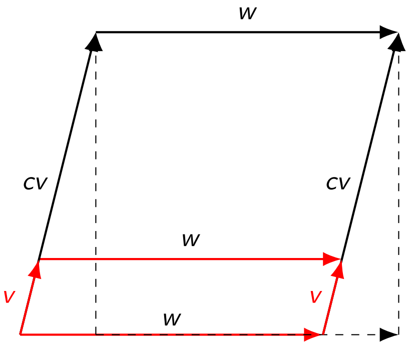
- If v_1 points upward and v_2 points downward relative to w, then $A(w, v_2) < 0$ and

$$A(w, v_1) = A(w, v_1 + v_2) - A(w, v_2)$$

and therefore

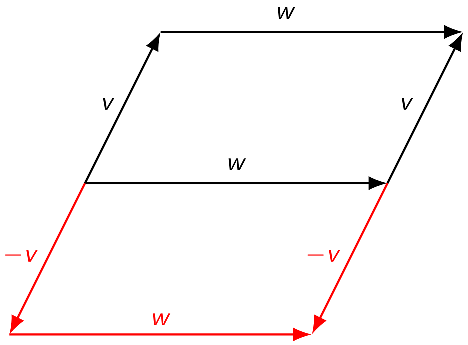
$$A(w, v_1 + v_2) = A(w, v_1) + A(w, v_2)$$

Area of rescaled parallelogram



$$A(w, cv) = cA(w, v)$$

Area of reflected parallelogram



$$A(w, -v) = A(w, v)$$

Area Versus Oriented Area

- Definitions of area and oriented area require a basis (e_1, e_2), where we assume that

$$A(e_1, e_2) = 1$$

- The oriented area of a parallelogram satisfies

$$\begin{aligned} A(v, w) &= -A(w, v) \\ A(v_1 + v_2, w) &= A(v_1, w) + A(v_2, w) \\ A(cv_1, v_2) &= cA(v_1, v_2) \end{aligned}$$

- The area of the parallelogram $P(v, w)$ is $|A(v, w)|$

Oriented Area is Bilinear and Antisymmetric

- If w is held fixed, $A(v, w)$ is a linear function of v

$$A(v_1 + v_2, w) = A(v_1, w) + A(v_2, w)$$
$$A(cv, w) = cA(v, w)$$

- If v is held fixed, $A(v, w)$ is a linear function of w

$$A(v, w_1 + w_2) = A(v, w_1) + A(v, w_2)$$
$$A(v, cw) = cA(v, w)$$

- Such a function is called **bilinear**
- For any $v \in V$, the parallelogram $A(v, v)$ has height 0 and therefore

$$A(v, v) = 0 \tag{1}$$

- Any bilinear function $A : V \times V \to \mathbb{F}$ that satisfies (1) is called **antisymmmetric**
- If A is antisymmetric and bilinear, then for any $v, w \in V$,

$$A(w, v) = -A(v, w)$$

2-Dimensional Antisymmetric Bilinear Function

- Let $\begin{bmatrix} e_1 e_2 \end{bmatrix}$ be a basis of V
- Let

$$A : V \times V \to \mathbb{F}$$

  be an antisymmetric bilinear function such that

$$A(e_1, e_2) = 1$$

- If $v = ae_1 + be_2$ and $w = ce_1 + de_2$, then

$$\begin{aligned} A(v, w) &= A(ae_1 + be_2, ce_1 + de_2) \\ &= A(ae_1, ce_1) + A(be_2, ce_1) + A(ae_1, de_2) + A(be_2, de_2) \\ &= bcA(e_2, e_1) + adA(e,e_2) \\ &= ad - bc \end{aligned}$$

2-Dimensional Antisymmetric Bilinear Function

- This can be written as follows

$$\begin{aligned} A\left(\begin{bmatrix} v & w \end{bmatrix}\right) &= A\left(\begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\ &= A\left(\begin{bmatrix} ae_1 + be_2 & ce_1 + de_2 \end{bmatrix}\right) \\ &= A(e_1, e_2)(ad - bc) \\ &= ad - bc \end{aligned}$$

- The determinant of a square 2-by-2 matrix is defined to be

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Determinant of a 2-by-2 Matrix is Equal to Oriented Area

- Let (e_1, e_2) be a basis where the oriented area of $P(e_1, e_2)$ is 1,

$$A(e_1, e_2) = 1$$

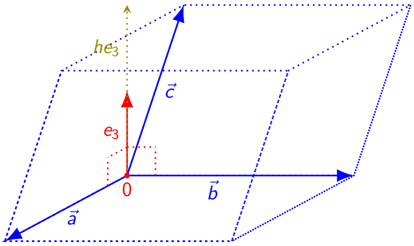
- The oriented area of the parallelogram $P(v, w)$, where

$$\begin{bmatrix} v & w \end{bmatrix} = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

  is

$$A(v, w) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

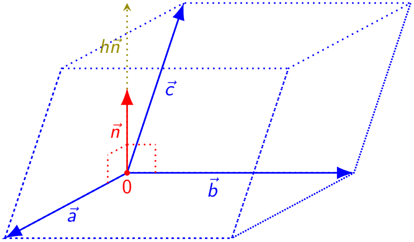
Parallelopiped spanned by 3 Vectors in 3-space



- Three linearly independent vectors $\vec{a}, \vec{b}, \vec{c}$ span a parallelopiped $P(\vec{a}, \vec{b}, \vec{c})$

$$P(\vec{a}, \vec{b}, \vec{c}) = \{s\vec{a} + t\vec{b} + u\vec{c} \ : \ 0 \leq s, t, u \leq 1\}$$

Volume of a Parallelopiped

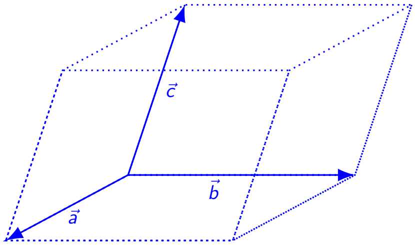


- Fix a basis (e_1, e_2, e_3) of V
  - Assume the volume of $P(e_1, e_2, e_2)$ is 1
- Assume $\vec{a}, \vec{b}$ lies in the subspace spanned by (e_1, e_2)
  - Therefore, $\vec{c} = he_3$
- If $h > 0$, then volume of parallelopiped is height times the area of the base:

$$\mathrm{vol}(P(\vec{a}, \vec{b}, \vec{c})) = h|A(\vec{a}, \vec{b})|$$

- Again, we want to avoid the absolute value

Oriented Volume of a Parallelopiped



- Define the oriented volume of $P\vec{a}, \vec{b}, \vec{c})$ to be

$$\text{vol}(\vec{a}, \vec{b}, \vec{c}),$$

  where
  - $\text{vol}(e_1, e_2, e_3) = 1$
  - $|\text{vol}(\vec{a}, \vec{b}, \vec{c})|$ is the volume of $P(\vec{a}, \vec{b}, \vec{c})$
  - vol is an antisymmetric multilinear function

Oriented Volume is Determinant of Matrix

- Suppose $v_1, v_2, v_3 \in V$, where, using Einstein notation,

$$\begin{aligned}\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} &= \begin{bmatrix} e_k A^k_1 & e_k A^k_2 & e_k A^k_3 \end{bmatrix} \\ &= \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} A^1_1 & A^1_2 & A^1_3 \\ A^2_1 & A^2_2 & A^2_3 \\ A^3_1 & A^3_2 & A^3_3 \end{bmatrix} \\ &= EA\end{aligned}$$

- The determinant of A is defined by the equation

$$\text{vol}(v_1, v_2, v_3) = E \det A$$

- In particular, since $\text{vol}(e_1, e_2, e_2) = 1$,

$$\det I = 1$$

Permutations

- A permutation is a bijective map $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$
- Let S_n be the set of all permutations of order n
- For any $\sigma_1, \sigma_2 \in S_n$,
$$\sigma_2 \circ \sigma_1 \in S_n$$
- For any $\sigma_1, \sigma_2, \sigma_3 \in S_n$,
$$(\sigma_3 \circ \sigma_2) \circ \sigma_1 = \sigma_3 \circ (\sigma_2 \circ \sigma 1)$$
- Let ι denote the identity map
- For any $\sigma \in S_n$,
$$\iota \circ \sigma = \sigma \circ \iota = \sigma$$
- Since σ is bijective, there exists a unique $\sigma^{-1} \in S_n$ such that
$$\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = \iota$$
- S_n is a group, where group multiplication is composition

Transpositions

- A transposition is a permutation τ that switches two elements and leaves the others unchanged.
- Example: $\tau : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$, where

$$\tau(1) = 1, \ \tau(2) = 4, \ \tau(3) = 3, \ \tau(4) = 2$$

- For any $1 \leq j, k \leq n$, let τ_{jk} be the transposition where for any $1 \leq i \leq S_n$,

$$\tau_{jk}(i) = \begin{cases} k & \text{if } i = j \\ j & \text{if } i = k \\ i & \text{if } i \neq j, k \end{cases}$$

- Observe that

$$\tau \circ \tau = \iota$$

and therefore

$$\tau^{-1} = \tau$$

Any permutation is a composition of transpositions

- Given $\sigma \in S_n$, denote $\sigma_0 = \sigma$
- If $\sigma_1 = \tau_{1,\sigma_0(1)} \circ \sigma_0$, then $\sigma_1(1) = 1$
- If $\sigma_2 = \tau_{2,\sigma_1(2)} \circ \sigma_1$, then

$$\sigma_2(1) = 1,\ \sigma_2(2) = 2$$

- Given $1 \leq k < n$, assume that σ_k satisfies

$$\sigma_k(1) = 1,\ \sigma_k(2) = 2, \ldots,\ \sigma_k(k) = k$$

- If $\sigma_{k+1} = \tau_{k+1,\sigma_k(k+1)} \circ \sigma_k$, then

$$\sigma_{k+1}(1) = 1,\ \sigma_{k+1}(2) = 2, \ldots,\ \sigma_{k+1}(k+1) = k+1$$

- By induction,

$$\tau_{n,\sigma_{n-1}(n)} \circ \tau_{n-1,\sigma_{n-2}(n-1)} \circ \cdots \circ \tau_{1,\sigma_0(1)} \circ \sigma_0 = \iota$$

and therefore

$$\sigma = \tau_{1,\sigma_0(1)} \circ \tau_2 \circ \cdots \circ \tau_{n,\sigma_{n-1}(n)}$$

Parity or Sign of a Permutation

- If $j \neq k$, call $\tau_{j,k}$ a **nontrivial transposition**
- Given any permutation $\sigma \in S_n$, its parity or sign, which we will write as $\epsilon(\sigma)$, is defined to be
  - 1 if σ is the composition of an even number of transpositions
  - -1 if σ is the composition of an odd number of transpositions
- Easy consequences
  - $\epsilon(\iota) = 1$
  - If $1 \leq j \neq k \leq n$, then $\epsilon(\tau_{j,k}) = -1$
  - For any $\sigma, \tau \in S_n$, $\epsilon(\sigma \circ \tau) = \epsilon(\sigma)\epsilon(\tau)$
  - For any $\sigma \in S_n$,
    $$\epsilon(\sigma^{-1}) = \epsilon(\sigma),$$
    because
    $$1 = \epsilon(\iota) = \epsilon(\sigma^{-1} \circ \sigma) = \epsilon(\sigma^{-1})\epsilon(\sigma)$$

Existence and Uniqueness of Sign Function

- This is the consequence of the following:
  - A permutation is never both the composition of an even number of transpositions and the composition of an odd number of transpositions
- There are straightforward elementary proofs
- There are also many sophisticated proofs

Endomorphisms of $\{1, \ldots, n\}$

- Let $\text{End}(n)$ denote the space of all maps
$$\phi : \{1, \ldots, n\} \to \{1, \ldots, n\}$$
- Let $\text{Aut}(n)$ denote the space of bijective maps in $\text{End}(n)$
- Observe that $S_n = \text{Aut}(n)$
- The sign function $\epsilon : S_n \to \{-1, 1\}$ can be extended to the function
$$\epsilon : \text{End}(n) \to \{-1, 0, 1\},$$
where, if $\phi \in S_n$, then $\epsilon(\phi)$ is defined as before and
$$\epsilon(\phi) = 0 \text{ if } \phi \notin S_n$$
- The extended sign function satisfies the following properties:
$$\begin{aligned}\epsilon(\sigma_1 \circ \sigma_2) &= \epsilon(\sigma_1)\epsilon(\sigma_2)\\ \epsilon(\iota) &= 1\\ \epsilon(\sigma) &= -1 \text{ if } \sigma \text{ is a nontrivial tranposition}\end{aligned}$$

Alternating Multilinear Functions

- Let V be an n-dimensional vector space
- Let $T : V \times \cdots \times V \to \mathbb{F}$ be a function of n vectors
- T is **multilinear** if for each $1 \leq k \leq n$, $v_1, \ldots, v_n, w_k \in V$, $a, b \in \mathbb{F}$,

$$T(v_1, \ldots, av_k + bw_k, \ldots, v_n) = aT(v_1, \ldots, v_k, \ldots, v_n) + bT(v_1, \ldots, w_k, \ldots, v_n)$$

- T is **alternating** if for any $v_1, \ldots, v_n \in V$ and $\sigma \in S_n$,

$$T(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \epsilon(\sigma) T(v_1, \ldots, v_n)$$

  or, equivalently, for any $v_1, \ldots, v_n \in V$ and $\sigma \in \mathrm{End}(n)$,

$$T(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \epsilon(\sigma) T(v_1, \ldots, v_n)$$

- Let $\Lambda^n V^*$ denote the set of all alternating multilinear functions on V
- Each $T \in \Lambda^n V^* \backslash \{0\}$ is also called an **oriented volume function** of V