PULLBACK OF A VECTOR BUNDLE AND CONNECTION

DEANE YANG

Everything is assumed to be C^∞.

1. Pullback of a vector bundle

Let E be a rank r vector bundle over an m-manifold M. Given an n-manifold N and a smooth map $\Phi : N \to M$, the pullback bundle E^Φ over N is defined as follows:

- The fiber at $x \in N$ is $E^\Phi_x = E_{\Phi(x)}$.
- Given any local frame $(e_1, \dots, e_r)$ of E, $(e_1 \circ \Phi, \cdots, e_r \circ \Phi)$ is a local frame of E^Φ. In particular, any section f of E^Φ can be written locally as
$$f(x) = a^k(x) e_k(\Phi(x)),$$
where $(e_1, \dots, e_r)$ is a local frame of E and $(a^1, \dots, a^r) : N \to \mathbb{R}^r$ is smooth.

In other words, the space of smooth sections of E^Φ is the $C^\infty(N)$-module generated by the space of smooth sections of E pulled back by Φ.

Given any section f of E, $f \circ \Phi$ is a section of E^Φ. For convenience, we will also denote $f \circ \Phi$ by simply f.

2. Pushforward of the Lie bracket

Using the pullback of the tangent bundle of M, we can derive a formula for the pushforward of the Lie bracket of two vector fields on N.

Let $(x^1, \dots, x^n)$ be local coordinates on N, and the corresponding coordinate vector fields by $(\partial_1, \dots, \partial_n)$. Let $(y^1, \dots, y^m)$ be local coordinates on M and $(Y_1, \dots, Y_m)$ be the corresponding coordinate vector fields. If $\Phi(x) = (y^1(x)), \dots, y^m(x))$, then, for any $1 \le p \le n$ and $1 \le j \le m$,
$$\Phi_* \partial_p = \phi^j_p Y_j$$

where
$$\phi^j_p = \partial_p y^j.$$
Therefore, for any $1 \le p, q \le n$ and $1 \le j \le m$,
$$\partial_p \phi^j_q = \partial_p \partial_q y^j = \partial_q \partial_p y^j = \partial_q \phi^j_p. \tag{1}$$

Given $V \in T_x N$ and a section $\widehat{W} = b^k Y_k$ of the pullback $T^\Phi_* N$ of the tangent bundle $T_* N$, we can define the directional derivative of $\widehat{W}$ at x to be
$$V\widehat{W} = \langle V, db^k \rangle Y_k.$$

The following lemma provides a formula for $\Phi_*[V, W]$ using the pullback bundle $T^\Phi_* M$.

Received by the editors April 1, 2021.

Lemma 1. *Given tangent vector fields V and W on N,*

$$\Phi_*[V,W] = V\Phi_*W - W\Phi_*V. \tag{2}$$

Proof. If $V = a^p\partial_p$ and $W = b^p\partial_p$, then

$$\begin{aligned}
V\Phi_*W - W\Phi_*V &= a^p\partial_p(\phi^k_q b^q)Y_k - b^p\partial_p(\phi^k_q a^q)Y_k \\
&= (a^p\partial_p b^q - b^p\partial_p a^q)\phi^k_q Y_k + a^p b^q(\partial_p\phi^k_q - \partial_q\phi^k_p))Y_k \\
&= (a^p\partial_p b^q - b^p\partial_p a^q)\Phi_*\partial_q \\
&= \Phi_*[V,W].
\end{aligned}$$

□

If the pushforwards Φ_*V and Φ_*W can be extended smooth vector fields $\widetilde{V}$ and $\widetilde{W}$ on M, then they are said to be Φ-related. In that case, the left side of the equation below is equal to the right side of (2), and therefore

$$[\Phi_*V, \Phi_*W] = \Phi_*[V,W].$$

follows from the lemma. In particular, this is the case, if Φ is an embedding.

Otherwise, the left side of this identity is ill-defined and therefore the identity does not hold in general.

3. Pullback of a connection

Let ∇ be a connection on E. Observe that, since each e_k is a section on M, given any vector field v on M, $(\nabla_v e_k)$ is a section of E.

The pullback connection ∇^Φ is a connection on E^Φ that is defined as follows: Given any $V \in T_xN$,

$$\nabla^\Phi_V e_k = \nabla_{\Phi_*V} e_k.$$

Therefore, given a smooth section $f = a^k e_k$ of E^Φ,

$$\begin{aligned}
\nabla^\Phi_V f(x) &= \nabla^\Phi_V(a^k e_k) \\
&= \langle V, da^k(x)\rangle e_k + a^k\nabla_{\Phi_*V} e_k.
\end{aligned}$$

Observe that if V is a tangent vector field on N, then $\nabla^\Phi_V f$ is a smooth section of E^Φ.

4. Pullback of curvature

Given a section $f = a^k e_k$ of E^Φ,

$$\begin{aligned}
\nabla^\Phi_{\partial_p}(\nabla^\Phi_{\partial_q} f) &= \nabla^\Phi_{\partial_p}(\partial_q a^k e_k + a^k\nabla_{\Phi_*\partial_q} e_k) \\
&= \nabla^\Phi_{\partial_p}(\partial_q a^k e_k + a^k\phi^j_q\nabla_{Y_j} e_k) \\
&= \partial_p(\partial_q a^k)e_k + \partial_q a^k\nabla_{\Phi_*\partial_p} e_k + \partial_p(a^k\phi^j_q)\nabla_{Y_j} e_k + a^k\phi^j_q\nabla_{\Phi_*\partial_p}(\nabla_{Y_j} e_k) \\
&= \partial_p(\partial_q a^k)e_k + \phi^j_p\partial_q a^k\nabla_{Y_j} e_k + \partial_p(a^k\phi^j_q)\nabla_{Y_j} e_k + a^k\phi^j_q\phi^i_p\nabla_{Y_i}(\nabla_{Y_j} e_k)
\end{aligned}$$

Therefore, since $[\partial_p, \partial_q] = 0$ and $[Y_i, Y_j] = 0$, if R is the curvature tensor of ∇,

$$\begin{aligned}\nabla^\Phi_{\partial_p}(\nabla^\Phi_{\partial_q} f) - \nabla^\Phi_{\partial_q}(\nabla^\Phi_{\partial_p} f) &= a^k \phi^i_p \phi^j_q (\nabla_{Y_i}(\nabla_{Y_j} e_k) - \nabla_{Y_j}(\nabla_{Y_i} e_k)) \\ &= a^k \phi^i_p \phi^j_q R(Y_i, Y_j) e_k \\ &= R(\phi^i_p Y_i, \phi^j_q Y_j) a^k e_k \\ &= R(\Phi_* \partial_p, \Phi_* \partial_q) f.\end{aligned}$$

Therefore, if R^Φ is the curvature tensor of ∇^Φ, then, for any $x \in N$ and $f \in E^\Phi_x = E_{\Phi(x)}$,

$$R^\Phi(\partial_p, \partial_q) f = R(\Phi_* \partial_p, \Phi_* \partial_q) f.$$

From this, a straightforward calculation shows that given any vector fields V, W on N and section f of E^Φ,

$$R^\Phi(V, W) f = \nabla^\Phi_V(\nabla^\Phi_W f) - \nabla^\Phi_W(\nabla^\Phi_V f) - \nabla^\Phi_{[V,W]} f = R(\Phi_* V, \Phi_* W) f. \tag{3}$$

5. Pullback of torsion-free connection

Assume that ∇ is a torsion-free connection on $T_* M$, i.e.,

$$\nabla_{Y_j} Y_k = \nabla_{Y_k} Y_j. \tag{4}$$

By the definition of a pullback connection in §3,

$$\begin{aligned}\nabla^\Phi_{\partial_p} \Phi_* \partial_q &= \nabla^\Phi_{\partial_p}(\phi^j_q Y_j) \\ &= (\partial_p \phi^j_q) Y_j + \phi^j_q \nabla_{\Phi_* \partial_p} Y_j \\ &= (\partial_p \phi^j_q) Y_j + \phi^j_q \phi^k_p \nabla_{Y_k} Y_j.\end{aligned}$$

Similarly,

$$\nabla^\Phi_{\partial_q} \Phi_* \partial_p = (\partial_q \phi^j_p) Y_j + \phi^j_p \phi^k_q \nabla_{Y_k} Y_j.$$

Therefore, by (1) and (4),

$$\begin{aligned}\nabla^\Phi_{\partial_p} \Phi_* \partial_q - \nabla^\Phi_{\partial_q} \Phi_* \partial_p &= (\partial_p \phi^j_q - \partial_q \phi^j_p) Y_j + \phi^j_q \phi^k_p (\nabla_{Y_k} Y_j - \nabla_{Y_j} Y_k) \\ &= 0.\end{aligned} \tag{5}$$

This and the properties of a connection imply the following:

Lemma 2. *Given a smooth map $\Phi : N \to M$, and tangent vector fields V, W on N,*

$$\nabla^\Phi_V \Phi_* W - \nabla^\Phi_W \Phi_* V = \Phi_*[V, W].$$

Lemma 1 corresponds to the case where the connection on M is the flat connection, where $Y^1, \dots, Y^m$ are all constant with respect to the connection.

6. Riemann curvature

If ∇ is the Levi-Civita connection and R is the Riemannian curvature of a Riemannian metric on M, then for any smooth tangent vector fields U, V, W on N, then by (3), the curvature of ∇^Φ is given by

$$R^\Phi(V, W) U = \nabla^\Phi_V(\nabla^\Phi_W U) - \nabla^\Phi_W(\nabla^\Phi_V U) - \nabla^\Phi_{[V,W]} U = R(\Phi_* V, \Phi_* W) \Phi_* U.$$

Observe that this identity holds for any smooth map Φ, and there are no assumptions about the dimensions of N and M. Also, no connection or Riemannian metric on N is needed.

7. Jacobi fields

A Jacobi field on a Riemannian manifold M is the variation of a 1-parameter family of geodesics, which is a map $\Phi : I_1 \times I_2 \to M$, where I_1, I_2 are intervals. Since the vector fields ∂_1, ∂_2 are not necessarily Φ-releated, Lemma 2 plays a crucial role in calculations involving Jacobi fields, including the proof that a Jacobi field satisfies the Jacobi equation. In most expositions of Jacobi fields, this is done in an ad hoc manner. This article shows that it fits within the more general framework of the pullback of the tangent bundle and Levi-Civita connection.

Department of Mathematics, New York University