SHARP AFFINE L_p SOBOLEV INEQUALITIES

Erwin Lutwak, Deane Yang and Gaoyong Zhang

0. Introduction

In this paper we prove a sharp affine L_p Sobolev inequality for functions on $\mathbb{R}^n$. The new inequality is significantly stronger than (and directly implies) the classical sharp L_p Sobolev inequality of Aubin [A2] and Talenti [T], even though it uses only the vector space structure and standard Lebesgue measure on $\mathbb{R}^n$. For the new inequality, no inner product, norm, or conformal structure is needed at all. In other words, the inequality is invariant under all affine transformations of $\mathbb{R}^n$. That such an inequality exists is surprising because the classical sharp L_p Sobolev inequality relies strongly on the Euclidean geometric structure of $\mathbb{R}^n$, especially on the isoperimetric inequality.

Zhang [Z] formulated and proved the sharp affine L_1 Sobolev inequality and established its equivalence to an L_1 affine isoperimetric inequality that is also proved in [Z]. He also showed that the affine L_1 Sobolev inequality is stronger than the classical L_1 Sobolev inequality.

The L_1 Sobolev inequality is known to be equivalent to the isoperimetric inequality (see, for example, [F], [FF], [M], [BZ], [O], and [SY]). The geometry behind the sharp L_p Sobolev inequality is also the isoperimetric inequality. For the affine Sobolev inequalities the situation is quite different. The geometric inequality and the critical tools used to establish the affine L_1 Sobolev inequality are not strong enough to enable us to establish the affine L_p Sobolev inequality for $p > 1$. A new geometric inequality and new tools are needed. The inequality needed is an affine L_p affine isoperimetric inequality recently established by the authors in [LYZ1] (see Campi and Gronchi [CG] for a recent alternate approach). We will also need the solution of an L_p extension of the classical Minkowski problem obtained in [L2]. It is crucial to observe that while the geometric core of the classical L_p Sobolev inequality (i.e., the isoperimetric inequality) is the same for all p, the geometric inequality (i.e., the affine L_p isoperimetric inequality) behind the new affine L_p Sobolev inequality is different for different p.

Let $\mathbb{R}^n$ denote n–dimensional Euclidean space; throughout we will assume that $n \geq 2$. Let $H^{1,p}(\mathbb{R}^n)$ denote the usual Sobolev space of real-valued functions of $\mathbb{R}^n$ with L_p partial derivatives.

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The classical sharp L_p Sobolev inequality of Aubin [A2] and Talenti [T] states that if $f \in H^{1,p}(\mathbb{R}^n)$, with real p satisfying $1 < p < n$, and if q is given by $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$, then

$$\left(\int_{\mathbb{R}^n} |\nabla f|^p \, dx \right)^{\frac{1}{p}} \geq c_0 \, \|f\|_q, \tag{0.1}$$

where $|\nabla f|$ is the Euclidean norm of the gradient of f, while $\|f\|_q$ is the usual L_q norm of f in $\mathbb{R}^n$, and

$$c_0 = n^{\frac{1}{p}} \left(\tfrac{n-p}{p-1} \right)^{1-\frac{1}{p}} \left[\omega_n \Gamma(\tfrac{n}{p}) \Gamma(n+1-\tfrac{n}{p}) / \Gamma(n) \right]^{\frac{1}{n}},$$

where ω_n is the n-dimensional volume enclosed by the unit sphere S^{n-1} in $\mathbb{R}^n$. Generalizations and related problems have been much studied (see, e.g., [AL], [BL], [Be], [BP], [BH], [BN], [CL], [CC], [D], [HV], [HS], [LZ], [Lie], [Y], [Z], and the references therein).

Since the case $p = 1$ of the sharp affine L_p Sobolev inequality was settled in [Z], in this paper we will focus exclusively on the case $p > 1$.

The basic concept behind our new inequality is a Banach space that we will associate with each function in $H^{1,p}(\mathbb{R}^n)$. The critical observation here is that this association is *affine* in nature. For real $p \geq 1$, we associate with each $f \in H^{1,p}(\mathbb{R}^n)$ a Banach norm $\| \cdot \|_{f,p}$ on $\mathbb{R}^n$. For $v \in S^{n-1}$ define

$$\|v\|_{f,p} = \|D_v f\|_p = \left(\int_{\mathbb{R}^n} |D_v f(x)|^p dx \right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^n} |v \cdot \nabla f(x)|^p dx \right)^{\frac{1}{p}},$$

where $D_v f$ is the directional derivative of f in the direction v. The integral on the right immediately provides the extension of $\| \cdot \|_{f,p}$ from S^{n-1} to $\mathbb{R}^n$. Now $(\mathbb{R}^n, \| \cdot \|_{f,p})$ is the n-dimensional Banach space that we shall associate with f. Its unit ball $B_p(f) = \{v \in \mathbb{R}^n : \|v\|_{f,p} \leq 1\}$ is a symmetric convex body in $\mathbb{R}^n$ and our new inequality states that the volume of this unit ball, $|B_p(f)|$, can be bounded from above by the reciprocal of the ordinary L_q-norm of f. Specifically, we have:

Theorem 1. *Suppose $p \in (1, n)$ and q is given by $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. If $f \in H^{1,p}(\mathbb{R}^n)$, then*

$$|B_p(f)|^{1/n} \leq c_1 / \|f\|_q, \tag{0.2}$$

where the best possible c_1 is given by

$$c_1 = \left(\tfrac{p-1}{n-p} \right)^{1-\frac{1}{p}} \left(\tfrac{\Gamma(n)}{\Gamma(\frac{n}{p}) \Gamma(n+1-\frac{n}{p})} \right)^{\frac{1}{n}} \left(\tfrac{\sqrt{\pi}\, \Gamma(\frac{n+p}{2})}{n \Gamma(\frac{n}{2}) \Gamma(\frac{p+1}{2})} \right)^{\frac{1}{p}}$$

and equality is attained when

$$f(x) = (a + |A(x - x_0)|^{\frac{p}{p-1}})^{1-\frac{n}{p}},$$

with $A \in GL(n)$, real $a > 0$, and $x_0 \in \mathbb{R}^n$.

Since the volume of the symmetric convex body $B_p(f)$ is obviously given by

$$|B_p(f)| = \frac{1}{n} \int_{S^{n-1}} \|D_v f\|_p^{-n} dv,$$

we can rewrite our main theorem as the following affine L_p Sobolev inequality:

Theorem 1′. *Suppose $p \in (1,n)$ and q is given by $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. If $f \in H^{1,p}(\mathbb{R}^n)$, then*

$$\left(\int_{S^{n-1}} \|D_v f\|_p^{-n} dv \right)^{-1/n} \geq c_2 \|f\|_q, \tag{0.3}$$

where the best possible c_2 is given by

$$c_2 = \left(\tfrac{n-p}{p-1}\right)^{1-\frac{1}{p}} \left(\tfrac{\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})}{\Gamma(n+1)}\right)^{\frac{1}{n}} \left(\tfrac{n\Gamma(\frac{n}{2})\Gamma(\frac{p+1}{2})}{\sqrt{\pi}\Gamma(\frac{n+p}{2})}\right)^{\frac{1}{p}}$$

and equality is attained when

$$f(x) = (a + |A(x - x_0)|^{\frac{p}{p-1}})^{1-\frac{n}{p}},$$

with $A \in GL(n)$, real $a > 0$, and $x_0 \in \mathbb{R}^n$.

Using the obvious fact that

$$\frac{1}{n!} \int_{\mathbb{R}^n} e^{-\|D_v f\|_p} dv = \frac{1}{n} \int_{S^{n-1}} \|D_v f\|_p^{-n} dv,$$

we can in turn rewrite Theorem 1′ as:

Theorem 1″. *Suppose $p \in (1,n)$ and q is given by $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. If $f \in H^{1,p}(\mathbb{R}^n)$, then*

$$\left(\int_{\mathbb{R}^n} e^{-\|D_v f\|_p} dv \right)^{-\frac{1}{n}} \geq c_3 \|f\|_q, \tag{0.4}$$

where the best possible c_3 is given by

$$c_3 = \left(\tfrac{n-p}{p-1}\right)^{1-\frac{1}{p}} \left(\tfrac{\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})}{\Gamma(n)\Gamma(n+1)}\right)^{\frac{1}{n}} \left(\tfrac{n\Gamma(\frac{n}{2})\Gamma(\frac{p+1}{2})}{\sqrt{\pi}\Gamma(\frac{n+p}{2})}\right)^{\frac{1}{p}},$$

and equality is attained when

$$f(x) = (a + |A(x - x_0)|^{\frac{p}{p-1}})^{1-\frac{n}{p}},$$

with $A \in GL(n)$, real $a > 0$, and $x_0 \in \mathbb{R}^n$.

Observe that inequality (0.4), and thus also inequality (0.3), is invariant under affine transformations of $\mathbb{R}^n$, while the L_p Sobolev inequality (0.1) is invariant only under rigid motions.

That the affine L_p Sobolev inequality (0.3) or (0.4) is stronger than the classical L_p Sobolev inequality (0.1) follows directly from the Hölder inequality, as will be shown in Section 7. We also note that the affine L_2 Sobolev inequality and the classical L_2 Sobolev inequality are equivalent under an affine transformation since the L_2 Banach norm $\|\cdot\|_{f,2}$ is Euclidean.

In Section 8, we present an application of the affine L_p Sobolev inequality to information theory. For a random vector X in a finite dimensional Banach space that is associated to a function f, we prove a sharp inequality that gives the best lower bound of the moments of X with respect to the Banach norm in terms of the λ-Renýi entropy of X and the L_q norm of f. Additional applications will be given in a forthcoming paper.

1. Background

For quick reference we list some facts about convex bodies. See [G], [S] and [Th] for additional details. A *convex body* is a compact convex set in $\mathbb{R}^n$ with nonempty interior. In this paper it will always be assumed that a convex body contains the origin in its interior. A convex body K is uniquely determined by its *support function* $h(K, \cdot\,) = h_K : \mathbb{R}^n \to (0, \infty)$, defined for $v \in \mathbb{R}^n$ by

$$h_K(v) = \max\{v \cdot x : x \in K\},$$

where $v \cdot x$ denotes the usual inner product of v and x in $\mathbb{R}^n$. The n-dimensional volume of K will be denoted by $V(K)$ or $|K|$.

For real $p \geq 1$, convex bodies K, L and real $\varepsilon > 0$, the *Minkowski-Firey L_p combination*, $K +_p \varepsilon \cdot L$, is the convex body whose support function is given

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \,\cdot\,)^p.$$

The *L_p-mixed volume* $V_p(K, L)$ of convex bodies K and L is defined by

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

Note that $V_p(Q, Q) = V(Q)$ for each convex body Q. It was shown in [L2] that there exists a unique finite positive Borel measure $S_p(K, \cdot\,)$ on S^{n-1} such that

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h_Q(v)^p dS_p(K, v), \tag{1.1}$$

for each convex body Q. The measure $S_p(K, \cdot\,)$ is called the *L_p-surface area measure* of K. The measure $S_1(K, \cdot\,) = S_K$ is the classical surface area measure of K. It was shown in [L2] that the measure $S_p(K, \cdot\,)$ is absolutely continuous with respect to S_K and the Radon-Nikodym derivative

$$\frac{dS_p(K, \,\cdot\,)}{dS_K} = h_K^{1-p}.$$

If boundary, ∂K, of K is C^2 with positive curvature, then the Radon-Nikodym derivative of S_K with respect to the Lebesgue measure on S^{n-1} is the reciprocal of the Gauss curvature of ∂K (when viewed as a function of the outer normals of ∂K).

A *compact domain* is the closure of a bounded open set. For compact domains M_1, M_2 and real $\lambda_1, \lambda_1 \geq 0$, the Minkowski linear combination $\lambda_1 M_1 + \lambda_2 M_2$ is defined by

$$\lambda_1 M_1 + \lambda_2 M_2 = \{\lambda_1 x_1 + \lambda_2 x_2 : x_1 \in M_1 \text{ and } x_2 \in M_2\}.$$

The *Brunn-Minkowski inequality* states that if M_1, M_2 are compact domains in $\mathbb{R}^n$ and $\lambda_1, \lambda_1 \geq 0$, then

$$V(\lambda_1 M_1 + \lambda_2 M_2)^{1/n} \geq \lambda_1 V(M_1)^{1/n} + \lambda_2 V(M_2)^{1/n},$$

where V denotes n-dimensional volume. If M is a compact domain and K is a convex body in $\mathbb{R}^n$ define the mixed volume, $V_1(M,K)$, of M and K by

$$nV_1(M,K) = \liminf_{\varepsilon\to 0^+} \frac{V(M+\varepsilon K) - V(M)}{\varepsilon}.$$

We shall require the following *Minkowski mixed volume inequality* for compact domains: If M is a compact domain in $\mathbb{R}^n$ and K is a convex body in $\mathbb{R}^n$, then

$$V_1(M,K)^n \geq V(M)^{n-1}V(K). \tag{1.2}$$

Note that (1.2) follows immediately from the definition of mixed volumes and the Brunn-Minkowski inequality:

$$\begin{aligned} V_1(M,K) &= \frac{1}{n}\liminf_{\varepsilon\to 0^+} \frac{V(M+\varepsilon K) - V(M)}{\varepsilon} \\ &\geq \frac{1}{n}\lim_{\varepsilon\to 0^+} \frac{[V(M)^{1/n} + \varepsilon V(K)^{1/n}]^n - V(M)}{\varepsilon} \\ &= V(M)^{\frac{n-1}{n}} V(K)^{\frac{1}{n}}. \end{aligned}$$

We will also require the following integral representation: Suppose M is a compact domain with C^1 boundary, ∂M. Then, if K is a convex body in $\mathbb{R}^n$,

$$V_1(M,K) = \frac{1}{n}\int_{\partial M} h_K(\nu(x))dS_M(x), \tag{1.3}$$

where $\nu(x)$ denotes the exterior unit normal at $x \in \partial M$, and $dS_M(x)$ is the surface area element at $x \in \partial M$. Identity (1.3) can be found, e.g., in [Z].

2. Affine L_p isoperimetric inequalities

We require an L_p-affine isoperimetric inequality that was first proved in [LYZ1] (see Campi and Gronchi [CG] for an alternative proof and generalizations). This inequality is one of the key ingredients in the proof of Theorem 1. Special cases of this new inequality and their relations to other affine isoperimetric inequalities can be found in, e.g., [L1] and [Le].

While the L_p mixed volume $V_p(\,\cdot\,,\,\cdot\,)$ has been defined only for compact convex sets that contain the origin in their interiors, a simple continuity argument allows us to extend the definition of the L_p-mixed volume $V_p(K,L)$ to the case where K is a compact convex set that contains the origin in its interior and L is a compact convex set that contains the origin in its relative interior. For $u \in S^{n-1}$, let $\bar{u}$ denote the line segment connecting the points $-u/2$ to $u/2$. Note that from (1.1) we have

$$V_p(K,\bar{u}) = \frac{1}{2^p n}\int_{S^{n-1}} |v\cdot u|^p dS_p(K,v), \tag{2.1}$$

for each $u \in S^{n-1}$.

Let c_4 be defined by

$$c_4 = \frac{\sqrt{\pi}\,\Gamma(\frac{n+p}{2})}{\Gamma(\frac{n}{2}+1)\Gamma(\frac{p+1}{2})}.$$

The following affine isoperimetric inequality was established in [LYZ1] and will be critical in establishing the affine L_p Sobolev inequality:

Theorem 2.1. *If real $p \geq 1$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then*

$$\left(\int_{S^{n-1}} V_p(K,\bar{v})^{-\frac{n}{p}}\,dv\right)^{\frac{p}{n}} V(K)^{\frac{n-p}{n}} \leq 2^{p-1} n^{1+\frac{p}{n}} c_4, \tag{2.2}$$

with equality if and only if K is an ellipsoid.

As an aside, we note that the actual inequality presented in [LYZ1] relates the volume of a convex body to that of its *polar L_p projection body.* However, the polar coordinate formula for volume quickly shows the equivalence of (2.2) and the polar L_p projection inequality that was established in [LYZ1].

3. The L_p Minkowski problem

We shall construct a family of convex bodies from a given function by using the solution to the even L_p-Minkowski problem. This will allow us to use the affine isoperimetric inequality (2.2) to establish Theorem 1.

A Borel measure on S^{n-1} is said to be *even* if for each Borel set $\omega \subset S^{n-1}$ the measure of ω and $-\omega = \{-x : x \in \omega\}$ are equal. In [L2], the following solution to the even case of the L_p-Minkowski problem is given:

Theorem 3.1. *Suppose μ is an even positive measure on S^{n-1} that is not supported on a great hypersphere of S^{n-1}. Then for real $p \geq 1$ such that $p \neq n$ there exists a unique origin-symmetric convex body K in $\mathbb{R}^n$ whose L_p-surface area measure is μ; i.e.,*

$$\mu = S_p(K, \,\cdot\,).$$

We now define for functions (rather than bodies) the notions of L_p mixed volumes. Suppose $f \in H^{1,p}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. For each real $t > 0$, define the level set,

$$[f]_t = \{x \in \mathbb{R}^n : |f(x)| \geq t\}.$$

Then $[f]_t$ is compact for each $t > 0$. By Sard's theorem, for almost all $t > 0$, the boundary of the level set $\partial[f]_t$ is a C^1 submanifold with everywhere nonzero normal vector ∇f. Abbreviate the surface area element of $\partial[f]_t$ by dS_t. If Q is a compact convex set that contains the origin in its relative interior, then define the *L_p-mixed volume* $V_p(f,t,Q)$, by

$$V_p(f,t,Q) = \frac{1}{n}\int_{\partial[f]_t} h_Q(\nu(x))^p |\nabla f(x)|^{p-1} dS_t(x), \tag{3.1}$$

where $\nu(x) = \nabla f(x)/|\nabla f(x)|$ is the outer unit normal at $x \in \partial[f]_t$. In particular, when Q is the line segment joining the points $-v/2$ and $v/2$, we have

$$V_p(f,t,\bar{v}) = \frac{1}{2^p n}\int_{\partial[f]_t} |v \cdot \nabla f(x)|^p |\nabla f(x)|^{-1} dS_t(x), \tag{3.2}$$

for each $v \in S^{n-1}$.

The following lemma shows that, for each fixed real $p \geq 1$, there is a natural way to associate a family of convex bodies with a given function.

Lemma 3.2. *If $f \in H^{1,p}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, then for almost every $t > 0$, there exists an origin-symmetric convex body K_t whose volume is given by*

$$V(K_t) = V_p(f, t, K_t) \tag{3.3a}$$

and such that for all $v \in S^{n-1}$,

$$V_p(f, t, \bar{v}) = V_p(K_t, \bar{v}). \tag{3.3b}$$

Proof. Define the positive Borel measure μ_t on S^{n-1} by

$$\int_{S^{n-1}} g(v) d\mu_t(v) = \int_{\partial[f]_t} g(\nu(x)) |\nabla f(x)|^{p-1} dS_t(x), \tag{3.4}$$

for each continuous $g : S^{n-1} \to \mathbb{R}$. Define the even Borel measure μ_t^* on S^{n-1} by letting

$$\mu_t^*(\omega) = \tfrac{1}{2}\mu_t(\omega) + \tfrac{1}{2}\mu_t(-\omega),$$

for each Borel $\omega \subset S^{n-1}$. Obviously, for each continuous even $g : S^{n-1} \to \mathbb{R}$,

$$\int_{S^{n-1}} g(v)\, d\mu_t^*(v) = \int_{S^{n-1}} g(v)\, d\mu_t(v). \tag{3.5}$$

From (3.5), (3.4), and the fact that $[f]_t$ has non-empty interior, it follows that for each $u \in S^{n-1}$,

$$\int_{S^{n-1}} |u \cdot v| d\mu_t^*(v) = \int_{S^{n-1}} |u \cdot v| d\mu_t(v) = \int_{\partial[f]_t} |u \cdot \nu(x)| |\nabla f(x)|^{p-1} dS_t(x) > 0.$$

Hence, the measure μ_t^* is not supported on any great hypersphere of S^{n-1}. By Theorem 3.1, there exists a unique origin-symmetric convex body K_t so that

$$d\mu_t^* = dS_p(K_t, \ \cdot\ ) = h_{K_t}^{1-p} dS_{K_t}. \tag{3.6}$$

To see that for each origin-symmetric convex body Q,

$$V_p(K_t, Q) = V_p(f, t, Q), \tag{3.7}$$

note that from (1.1) and (3.6), (3.5), (3.4) and definition (3.1), it follows that

$$\begin{aligned} V_p(K_t, Q) &= \frac{1}{n} \int_{S^{n-1}} h_Q(v)^p \, d\mu_t^*(v) \\ &= \frac{1}{n} \int_{S^{n-1}} h_Q(v)^p \, d\mu_t(v) \\ &= \frac{1}{n} \int_{\partial[f]_t} h_Q(\nu(x))^p |\nabla f(x)|^{p-1} dS_t(x) \\ &= V_p(f, t, Q). \end{aligned}$$

Now (3.7) and a continuity argument immediately yields (3.3b). To get (3.3a) take $Q = K_t$ in (3.7) and recall that $V_p(Q, Q) = V(Q)$ □

4. An integral inequality

The following well-known (see, e.g., [A1]) consequence of Bliss' inequality [B] will be needed. For the sake of completeness we include an elementary proof that uses techniques similar to ones used in [CNV].

Lemma 4.1. *Let f be a nonnegative differentiable function in $(0,\infty)$, $q = \frac{np}{n-p}$, and $1 < p < n$. If the integrals exist, then*

$$\left(\int_0^\infty |f'(x)|^p x^{n-1} dx\right)^{\frac{1}{p}} \geq c_5 \left(\int_0^\infty f(x)^q x^{n-1} dx\right)^{\frac{1}{q}},$$

where

$$c_5 = n^{\frac{1}{q}} (\tfrac{n-p}{p-1})^{1-\frac{1}{p}} \left[\Gamma(\tfrac{n}{p})\Gamma(n+1-\tfrac{n}{p})/\Gamma(n)\right]^{\frac{1}{n}} = (n\omega_n)^{-\frac{1}{n}} c_0,$$

Equality holds if $f(x) = (ax^{\frac{p}{p-1}} + b)^{1-\frac{n}{p}}$, with $a, b > 0$.

Proof. It suffices to prove the inequality for a nonnegative compactly supported smooth function f satisfying

$$\int_0^\infty f(x)^q x^{n-1} dx = 1.$$

Let $f_0 : (0,\infty) \to [0,\infty)$ be a continuous function that is supported on a bounded interval $[0, R)$ for some $R > 0$ and that satisfies

$$\int_0^\infty f_0(x)^q x^{n-1} dx = \int_0^\infty f_0(x)^q x^{p^*+n-1} dx = 1,$$

where $p^* = \frac{p}{p-1}$. Define $y : [0,\infty) \to [0, R]$ by

$$\int_0^x f(s)^q s^{n-1} ds = \int_0^{y(x)} f_0(t)^q t^{n-1} dt.$$

It follows that

$$\begin{aligned} f_0(y)^{q-\frac{q}{n}} y^{n-1} y' &= f(x)^{q-\frac{q}{n}} x^{n-1} [(\frac{y}{x})^{n-1} y']^{\frac{1}{n}} \\ &\leq \frac{1}{n} f(x)^{q-\frac{q}{n}} x^{n-1} ((n-1)\frac{y}{x} + y') \\ &= \frac{1}{n} f(x)^{q-\frac{q}{n}} (x^{n-1} y)'. \end{aligned} \tag{4.1}$$

Equality in the inequality holds if and only if $y = \lambda x$, $\lambda > 0$.

Integration by parts and the Hölder inequality, will give

$$\begin{aligned} \int_0^\infty f(x)^{q-\frac{q}{n}} (x^{n-1} y)' dx &= -(q - \frac{q}{n}) \int_0^\infty f(x)^{q-\frac{q}{n}-1} f'(x) x^{n-1} y \, dx \\ &\leq (q - \frac{q}{n}) \int_0^\infty f(x)^{q-\frac{q}{n}-1} |f'(x)| x^{n-1} y \, dx \\ &\leq (q - \frac{q}{n}) \left(\int_0^\infty y^{p^*} f^q x^{n-1} dx\right)^{\frac{1}{p^*}} \left(\int_0^\infty |f'|^p x^{n-1} dx\right)^{\frac{1}{p}} \\ &= (q - \frac{q}{n}) \left(\int_0^\infty f_0^q y^{p^*+n-1} dy\right)^{\frac{1}{p^*}} \left(\int_0^\infty |f'|^p x^{n-1} dx\right)^{\frac{1}{p}}. \end{aligned} \tag{4.2}$$

By (4.1) and (4.2),

$$\left(\int_0^\infty |f'|^p x^{n-1} dx\right)^{\frac{1}{p}} \geq \frac{n(n-p)}{p(n-1)} \int_0^\infty f_0^{q-q/n} y^{n-1} dy. \tag{4.3}$$

Since (4.3) holds for any compactly supported function f_0, it holds for any positive continuous function. Moreover, equality holds for (4.3) if $y = \lambda x$ and $f' = \beta f^{\frac{n}{n-p}} y^{\frac{1}{p-1}}$ for some constant β. Integrating this gives the extremal function. In particular, the desired inequality follows by setting $f_0(y) = (ay^{\frac{p}{p-1}} + b)^{1-\frac{n}{p}}$, where a and b are chosen so that f_0 satisfies the required normalizations. □

5. A lemma about rearrangements

For $f : \mathbb{R}^n \to \mathbb{R}$ and real $t > 0$, let

$$[f]_t = \{x \in \mathbb{R}^n : |f(x)| \geq t\},$$

denote the level sets of f. We always assume that our functions are such that the level sets $[f]_t$ are compact for all $t > 0$.

The decreasing *rearrangement*, $\bar{f}$, of $f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\bar{f}(x) = \inf\{t > 0 : V([f]_t) < \omega_n |x|^n\},$$

where $\omega_n |x|^n$ is the n-dimensional volume of the ball of radius $|x|$ in $\mathbb{R}^n$. The set $[\bar{f}]_t = \{x \in \mathbb{R}^n : \bar{f}(x) \geq t\}$ is a dilate of the unit ball, $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$, and its volume is equal to $V([f]_t)$; i.e.,

$$V([\bar{f}]_t) = V([f]_t).$$

The functions f and $\bar{f}$ are equimeasurable, and therefore for all $q \geq 1$

$$\|f\|_q = \|\bar{f}\|_q. \tag{5.1}$$

Note that since $\bar{f}(x)$ depends only on the magnitude, $|x|$, of x (and not on the direction of x), there exists an increasing function $\hat{f} : (0, \infty) \to \mathbb{R}$ defined by

$$\bar{f}(x) = \hat{f}(1/|x|). \tag{5.2}$$

Observe that provided f is sufficiently smooth, $\bar{f}(x) = t$ implies (by definition of $\bar{f}$) that $V([f]_t) = \omega_n |x|^n$, or equivalently, $\hat{f}(1/|x|) = t$ implies $V([f]_t) = \omega_n |x|^n$.

The following is needed in the proof of the main theorem:

Lemma 5.1. *If $f \in H^{1,p}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, then*

$$\int_0^\infty V([f]_t)^{\frac{p(n-1)}{n}}(-V'([f]_t))^{1-p}dt = \frac{n^{1-p}}{\omega_n^{\frac{p-n}{n}}}\int_0^\infty (\hat{f}'(s))^p s^{2p-n-1}ds, \tag{5.3}$$

and

$$\|\bar{f}\|_q^q = n\omega_n \int_0^\infty \hat{f}(s)^q s^{-n-1}ds. \tag{5.4}$$

Proof. Let $t = \hat{f}(s)$. Since $\hat{f}(1/|x|) = t$ implies $V([f]_t) = \omega_n |x|^n$,

$$V([f]_t) = s^{-n}\omega_n,$$

and hence

$$-V'([f]_t) = ns^{-n-1}\frac{ds}{dt}\omega_n.$$

It follows that

$$V([f]_t)^{\frac{p(n-1)}{n}}(-V'([f]_t))^{1-p}dt = n^{1-p}s^{2p-n-1}\left(\frac{ds}{dt}\right)^{-p} ds\, \omega_n^{1-\frac{p}{n}},$$

which gives (5.3). To get (5.4) simply rewrite the defining integral for $\|\bar{f}\|_q^q$ in polar coordinates. $\square$

6. Affine L_p Sobolev inequalities

Theorem 1′. *Suppose $p \in (0,1)$, and q is given by $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. If $f \in H^{1,p}(\mathbb{R}^n)$, then*

$$\left(\int_{S^{n-1}} \|D_v f\|_p^{-n} dv\right)^{-1/n} \geq c_2 \|f\|_q, \tag{6.1}$$

where the optimal c_2 is given by

$$c_2 = \left(\tfrac{n-p}{p-1}\right)^{1-\frac{1}{p}} \left(\tfrac{\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})}{\Gamma(n+1)}\right)^{\frac{1}{n}} \left(\tfrac{n\Gamma(\frac{n}{2})\Gamma(\frac{p+1}{2})}{\sqrt{\pi}\Gamma(\frac{n+p}{2})}\right)^{\frac{1}{p}} = (2/nc_4)^{\frac{1}{p}}(n\omega_n)^{-\frac{1}{n}}c_0$$

and equality is attained when

$$f(x) = (a + |A(x-x_0)|^{\frac{p}{p-1}})^{1-\frac{n}{p}},$$

with $A \in GL(n)$, real $a > 0$ and $x_0 \in \mathbb{R}^n$.

Proof. It suffices to prove the inequality for compactly supported $f \in C^\infty(\mathbb{R}^n)$. For $t > 0$, consider the level sets of f,

$$[f]_t = \{x \in \mathbb{R}^n : |f(x)| \geq t\}.$$

By Sard's theorem, for almost all $t > 0$ the boundary, $\partial[f]_t$, of the level set is a C^1 submanifold which has everywhere nonzero normal vector ∇f. Let dS_t denote the surface area element of $\partial[f]_t$. For $t > 0$, let K_t be the convex body constructed from f in Lemma 3.2.

We first need

$$\|D_v f\|_p^p = 2^p n \int_0^\infty V_p(K_t, \bar{v})dt. \tag{6.2}$$

To see this simply note that by rewriting the integral, using (3.2), and then using (3.3b) we have

$$\begin{aligned}
\|D_v f\|_p^p &= \int_{\mathbb{R}^n} |v \cdot \nabla f(x)|^p dx \\
&= \int_0^\infty \int_{\partial[f]_t} |v \cdot \nabla f(x)|^p |\nabla f(x)|^{-1} dS_t(x)\, dt \\
&= 2^p n \int_0^\infty V_p(f, t, \bar{v})dt \\
&= 2^p n \int_0^\infty V_p(K_t, \bar{v})dt.
\end{aligned}$$

We need the fact that

$$V(K_t)^{n-p} \geq V([f]_t)^{(n-1)p}(-n^{-1}V'([f]_t))^{n(1-p)}. \tag{6.3}$$

To see this, note that from (3.3a), definition (3.1), the Hölder inequality, definition (3.1) again, the extended Minkowski mixed volume inequality (1.2), and the co-area formula, we have

$$\begin{aligned}
V(K_t)^{\frac{n-p}{np}} &= V(K_t)^{-\frac{1}{n}} V_p(f, t, K_t)^{\frac{1}{p}} \\
&= V(K_t)^{-\frac{1}{n}} \left(\frac{1}{n} \int_{\partial[f]_t} h_{K_t}(\nu(x))^p |\nabla f(x)|^{p-1} dS_t(x) \right)^{\frac{1}{p}} \\
&\geq n^{-\frac{1}{p}} V(K_t)^{-\frac{1}{n}} \left(\int_{\partial[f]_t} |\nabla f|^{-1} dS_t \right)^{\frac{1-p}{p}} \int_{\partial[f]_t} h_{K_t}(\nu(x)) dS_t(x) \\
&= n^{1-\frac{1}{p}} V(K_t)^{-\frac{1}{n}} \left(\int_{\partial[f]_t} |\nabla f|^{-1} dS_t \right)^{\frac{1-p}{p}} V_1([f]_t, K_t) \\
&\geq n^{1-\frac{1}{p}} \left(\int_{\partial[f]_t} |\nabla f|^{-1} dS_t \right)^{\frac{1-p}{p}} V([f]_t)^{\frac{n-1}{n}} \\
&= n^{1-\frac{1}{p}} \left(-V'([f]_t)\right)^{\frac{1-p}{p}} V([f]_t)^{\frac{n-1}{n}}.
\end{aligned}$$

To complete the proof, observe that from (6.2), the Minkowski inequality for integrals, the affine inequality (2.2), (6.3), (5.3), (5.4), and (5.1),

$$\begin{aligned}\left(\int_{S^{n-1}} \|D_v f\|_p^{-n} dv\right)^{-\frac{p}{n}} &= 2^p n \left(\int_{S^{n-1}} \left(\int_0^\infty V_p(K_t, \bar{v})\, dt\right)^{-\frac{n}{p}} dv\right)^{-\frac{p}{n}} \\ &\geq 2^p n \int_0^\infty \left(\int_{S^{n-1}} V_p(K_t, \bar{v})^{-\frac{n}{p}} dv\right)^{-\frac{p}{n}} dt \\ &\geq \frac{2}{c_4 n^{p/n}} \int_0^\infty V(K_t)^{\frac{n-p}{n}} dt \\ &\geq \frac{2n^{p-1-\frac{p}{n}}}{c_4} \int_0^\infty V([f]_t)^{\frac{(n-1)p}{n}} (-V'([f]_t))^{1-p} dt \\ &= \frac{2n^{-\frac{p}{n}} \omega_n^{\frac{n-p}{n}}}{c_4} \int_0^\infty (\hat{f}'(s))^p s^{2p-n-1} ds \\ &\geq \frac{2}{nc_4} c(\hat{f})^p \|\bar{f}\|_q^p \\ &= \frac{2}{nc_4} c(\hat{f})^p \|f\|_q^p,\end{aligned}$$

where

$$c(\hat{f}) = \left(\int_0^\infty (\hat{f}'(s))^p s^{2p-n-1} ds\right)^{\frac{1}{p}} \left(\int_0^\infty \hat{f}(s)^q s^{-n-1} ds\right)^{-\frac{1}{q}}.$$

Make the substitution $t = 1/s$ and then define the function g by $g(t) = \hat{f}(1/t)$, to get

$$c(\hat{f}) = \left(\int_0^\infty |g'(t)|^p t^{n-1} dt\right)^{\frac{1}{p}} \left(\int_0^\infty g(t)^q t^{n-1} dt\right)^{-\frac{1}{q}},$$

(recall that $\hat{f}$ is increasing and thus g' is always negative). Lemma 4.1 gives,

$$c(\hat{f}) \geq c_5$$

and this proves the desired inequality. □

Remark. The affine L_p-Sobolev inequality (0.4) implies the affine L_p-isoperimetric inequality (2.2). This can be seen by taking

$$f(x) = \left(1 + \rho_K(x)^{\frac{p}{1-p}}\right)^{1-\frac{n}{p}},$$

where $\rho_K(x) = \max\{\lambda \geq 0 \ : \ \lambda x \in K\}$ denotes the *radial function* of K.

A simple calculation shows that

$$\|D_v f\|_p^p = c_6 V_p(K, \bar{v}),$$

where

$$c_6 = n 2^p \left(\tfrac{n-p}{p-1}\right)^{p-1} \Gamma(\tfrac{n}{p})\Gamma(n+1-\tfrac{n}{p})/\Gamma(n).$$

Therefore, (2.2) is one of the consequences of the new inequality (0.4).

7. The L_p and affine L_p Sobolev inequalities

We will show that the new affine L_p Sobolev inequality is indeed stronger (and directly implies) the sharp L_p Sobolev inequality.

First observe that

$$\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \|D_v f\|_p^{-n} dv\right)^{-\frac{p}{n}} \leq \frac{2}{nc_4}\int_{\mathbb{R}^n} |\nabla f(x)|^p dx. \tag{7.1}$$

To see this, note that from the Hölder inequality and Fubini's theorem we have

$$\begin{aligned}\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \|D_v f\|_p^{-n} dv\right)^{-\frac{p}{n}} &\leq \frac{1}{n\omega_n}\int_{S^{n-1}} \|D_v f\|_p^{p} dv \\ &= \frac{1}{n\omega_n}\int_{S^{n-1}}\int_{\mathbb{R}^n} |v\cdot\nabla f(x)|^p dx dv \\ &= \frac{1}{n\omega_n}\int_{\mathbb{R}^n}\int_{S^{n-1}} |v\cdot\nabla f(x)|^p dv dx \\ &= \frac{1}{n\omega_n}\int_{S^{n-1}} |u_0\cdot v|^p dv \int_{\mathbb{R}^n} |\nabla f(x)|^p dx, \\ &= \frac{2}{nc_4}\int_{\mathbb{R}^n} |\nabla f(x)|^p dx,\end{aligned}$$

where u_0 is any fixed unit vector.

Now combine (7.1) with the affine L_p Sobolev inequality (6.1) to get:

$$\left(\int_{\mathbb{R}^n} |\nabla f|^p\, dx\right)^{\frac{1}{p}} \geq [nc_4/2]^{1/p}\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \|D_v f\|_p^{-n} dv\right)^{-\frac{1}{n}} \geq c_0\,\|f\|_q. \tag{7.2}$$

From the equality conditions in the Hölder inequality it follows that equality in the left inequality in (7.2) occurs precisely when f is such that $\|D_v f\|_p$ is independent of $v \in S^{n-1}$.

8. An application to information theory

In this section we use Theorem 1 to prove a moment-entropy inequality for a Banach space–valued random variable X.

Let X be a random vector in $\mathbb{R}^n$ with probability density g. Given $\lambda > 0$, the λ-Renyi entropy power $N_\lambda(X)$ is defined by

$$N_\lambda(X) = \begin{cases} \|g\|_\lambda^{\frac{\lambda}{1-\lambda}} & \lambda \neq 1 \\ e^{-\int g\log g} & \lambda = 1. \end{cases}$$

Observe that $N_1(X)$ is the Shannon entropy power of X, and

$$\lim_{\lambda\to 1} N_\lambda(X) = N_1(X).$$

A random vector X in $\mathbb{R}^n$ with density function g is said to have finite r^{th}-*moment*, $r > 0$, if

$$\int_{\mathbb{R}^n} |x|^r g(x)dx < \infty.$$

If X is a random vector in $\mathbb{R}^n$ with finite r^{th}-moment, and K is an origin-symmetric convex body in $\mathbb{R}^n$, then the dual mixed volume $\tilde{V}_{-r}(X, K)$ was defined in [LYZ2] by

$$\tilde{V}_{-r}(X, K) = \frac{n+r}{n} \int_{\mathbb{R}^n} \|x\|_K^r g(x)\, dx, \tag{8.1}$$

where g is the density function of X and $\|\cdot\|_K$ is the norm of the n-dimensional Banach space whose unit ball is K; i.e., for $x \in \mathbb{R}^n$

$$\|x\|_K = \min\{\lambda > 0 : x \in \lambda K\}.$$

The following is a special case of the dual Minkowski inequality established in [LYZ2].

Lemma 8.1. *Suppose $r > 0$ and $\lambda > \frac{n}{n+r}$. If K is an origin-symmetric convex body in $\mathbb{R}^n$ and X is a random vector in $\mathbb{R}^n$ with finite r^{th}-moment, then*

$$\tilde{V}_{-r}(X, K)^{\frac{1}{r}} \geq c_7 [N_\lambda(X)/|K|\,]^{\frac{1}{n}},$$

where the best constant c_7 is given by

$$c_7 = \begin{cases} \left(1 - \frac{n(1-\lambda)}{r\lambda}\right)^{\frac{1}{n(1-\lambda)}} \left(\frac{\lambda - \frac{n}{n+r}}{1-\lambda}\right)^{-\frac{1}{r}} \left(\frac{n}{r} B\left(\frac{n}{r}, \frac{1}{1-\lambda} - \frac{n}{r}\right)\right)^{-\frac{1}{n}} & \lambda < 1, \\ \left(\frac{n+r}{re}\right)^{\frac{1}{r}} \Gamma\left(\frac{n}{r} + 1\right)^{-\frac{1}{n}} & \lambda = 1, \\ \left(1 + \frac{n(\lambda-1)}{r\lambda}\right)^{\frac{1}{n(1-\lambda)}} \left(\frac{\lambda - \frac{n}{n+r}}{\lambda - 1}\right)^{-\frac{1}{r}} \left(\frac{n}{r} B\left(\frac{n}{r}, \frac{\lambda}{\lambda-1}\right)\right)^{-\frac{1}{n}} & \lambda > 1. \end{cases}$$

Given a function $f \in H^{1,p}(\mathbb{R}^n)$, let $\|\cdot\|_{f,p}$ denote the associated Banach norm defined in the Introduction. If X is a random vector in the Banach space $(\mathbb{R}^n, \|\cdot\|_{f,p})$, the r^{th} *moment* of X is $E(\|X\|_{f,p}^r)$. The following theorem gives a sharp lower bound of the moment $E(\|X\|_{f,p}^r)$ in terms of the Renyi entropy power $N_\lambda(X)$ and the L_q norm $\|f\|_q$.

Theorem 8.2. *Suppose $1 \leq p < n$, $q = \frac{np}{n-p}$, $r > 0$, and $\lambda > \frac{n}{n+r}$. If $f \in H^{1,p}(\mathbb{R}^n)$ and X is a random vector in $\mathbb{R}^n$ with finite r^{th}-moment, then*

$$E(\|X\|_{f,p}^r) \geq c_8 N_\lambda(X)^{\frac{r}{n}} \|f\|_q^r,$$

where the best constant $c_8 = nc_7^r c_1^{-r}/(n+r)$.

Proof. Let $B_p(f)$ denote the unit ball associated with the norm $\|\cdot\|_{f,p}$. Let g be the density function of X. Note that inequality (0.3) holds when $p = 1$ (see [Z]), and thus inequality (0.2) holds when $p = 1$. From (8.1), Lemma 8.1, and (0.2), we have

$$\begin{aligned} E(\|X\|_{f,p}^r) &= \int_{\mathbb{R}^n} \|x\|_{f,p}^r \, g(x)\, dx \\ &= \frac{n}{n+r} \tilde{V}_{-r}(X, B_p(f)) \\ &\geq \frac{n}{n+r} c_7^r [N_\lambda(X)/|B_p(f)|\,]^{\frac{r}{n}} \\ &\geq \frac{n}{n+r} c_7^r [N_\lambda(X) c_1^{-n} \|f\|_q^n]^{\frac{r}{n}} \\ &= c_8 [N_\lambda(X) \|f\|_q^n]^{\frac{r}{n}}. \quad \square \end{aligned}$$

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