THE EXTERIOR DERIVATIVE VIA STOKES'S THEOREM

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The normal path to Stokes's theorem is to begin by defining first differential forms, the exterior derivative, and the integral of a differential form. The path then culminates with the statement and proof of Stokes's theorem.

We take a slightly different path here. First, differential forms and the integral of a differential form are defined. We then seek a higher dimensional version of the fundamental theorem of calculus for an integral over a rectangular domain. This is accomplished using the fundamental theorem of calculus itself. The result is a simple version of Stokes's theorem. The exterior derivative of a differential form appears as the integrand of the integral over the rectangular domain. It is therefore a consequence of Stokes's theorem, rather than an a priori definition.

It is, however, necessary to show that the exterior derivative is well defined, independent of the coordinates used. This is accomplished by showing that exterior differentiation commutes with pulling back the differential form by a smooth map from the range of a smooth map to its domain.

Finally, we use the same approach to prove Stokes's theorem on a simplex. This then proves Stokes's theorem on a smoothly triangulated manifold possibly with boundary.

For convenience, $O \subset \mathbb{R}^m$ will always denote a connected open set. The domain of integration will always be a compact subset of O with piecewise smooth boundary. All functions, maps, and differential forms are assumed to be smooth.

1. Orientation

The space of m-forms on $\mathbb{R}^m$ is a 1-dimensional vector space and therefore the set of nonzero ones has two connected components. An orientation on $\mathbb{R}^m$ is one of the two components. Given a non-zero m-form Θ, let $[\Theta]$ denote the orientation containing Θ.

We will always use the orientation $[dx^1 \wedge \cdots \wedge dx^m]$ on $\mathbb{R}^m$.

2. Integral of a differential form on $\mathbb{R}^m$

Any differential m-form Θ on a domain $O \subset \mathbb{R}^m$ can written as

$$\Theta = a\, dx^1 \wedge \cdots \wedge dx^m.$$

Written in this form, the integral of Θ over the compact domain C is defined to be

$$\int_C \Theta = \int_C a\, dx^1 \cdots dx^m,$$

where the right side is an iterated integral. In particular, the integral is independent of the order of integration.

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3. Pullback of a differential form by a smooth map

Given an open set $O' \subset \mathbb{R}^n$, a smooth map $\Phi : O \to O'$, and a differential m-form on O', the pullback of Θ by Φ is a differential m-form, denoted $\Phi^*\Theta$, on O, where, for each $x \in O$ and $v_1, \dots, v_m \in \mathbb{R}^m$,

$$(\Phi^*\Theta)(x)(v_1, \dots, v_m) = \Theta(\Phi(x))(d\Phi(x)v_1, \dots, d\Phi(x)v_m),$$

where $d\Phi$ denotes the differential of Φ.

4. Integral of a differential form on a parameterized submanifold

Given an open set $O' \subset \mathbb{R}^n$ and a smooth embedding $\Phi : D \to O'$, the integral of a differential m-form Θ on O' over the submanifold $S = \Phi(D)$ is defined to be

$$\int_S \Theta = \int_D \Phi^*\Theta,$$

using the standard orientation on $\mathbb{R}^m$. The integral is independent of the parameterization Φ.

5. Integration over a rectangle

Recall that given an open set $O \subset \mathbb{R}^2$ and a function $f : O \to \mathbb{R}$, its differential is given by

$$df(x,y) = \partial_x f(x,y)\,dx + \partial_y f(x,y)\,dy.$$

Given $\delta, \epsilon > 0$, let

$$R = \{(x,y) \in \mathbb{R}^2 \ : \ 0 \le x \le \delta,\ 0 \le y \le \epsilon\}.$$

We will use standard orientation on $\mathbb{R}^2$, which is given by $dx \wedge dy$, and orient the boundary of R counterclockwise. This is equivalent to saying that the orientation on each side of R is $[n \rfloor dx \wedge dy]$, where $n \in \mathbb{R}^2$ points outward.

Let $\Theta = a(x,y)\,dx + b(x,y)\,dy$ be a differential 1-form on R. Integrating it along ∂R, we get

$$\begin{aligned}
\int_{\partial R} \Theta &= \int_{x=0}^{x=\delta} a(x,0)\,dx + \int_{y=0}^{y=\epsilon} b(\delta,y)\,dy + \int_{x=\delta}^{x=0} a(x,\epsilon)\,dx + \int_{y=\epsilon}^{y=0} b(0,y)\,dy \\
&= -\int_{x=0}^{x=\delta} a(x,\epsilon) - a(x,0)\,dx + \int_{y=0}^{y=\epsilon} b(\delta,y) - b(0,y)\,dy \\
&= -\int_{x=0}^{x=\delta} \int_{y=0}^{y=\epsilon} \partial_y a(x,y)\,dy\,dx + \int_{y=0}^{y=\epsilon} \int_{x=0}^{x=\delta} \partial_x b(x,y)\,dx\,dy \\
&= \int_R (-\partial_y a(x,y) + \partial_x b(x,y))\,dx \wedge dy \\
&= \int_R (\partial_x a(x,y)\,dx + \partial_y a(x,y)\,dy) \wedge dx + (\partial_x b(x,y)\,dx + \partial_y b(x,y)\,dy) \wedge dy \\
&= \int_R da \wedge dx + db \wedge dy.
\end{aligned}$$

6. Integration over an m-dimensional rectangular region

Given $\delta_1, \dots, \delta_m > 0$, let

$$R = [0, \delta_1] \times \cdots \times [0, \delta_m] \subset \mathbb{R}^m.$$

For each $1 \le i \le m$, let F_i denote the face that lies in the hyperplane $x^i = 0$ and $F_i + \delta_i e_i$ the one lying in the hyperplane $x^i = \delta_i$.

Let $e_1, \dots, e_m$ be the standard basis, $dx^1, \dots, dx^m$ be the dual basis, and denote $dx = dx^1 \wedge \cdots dx^m$. For each $1 \le i \le m$, let

$$\widehat{dx^i} = e_i \rfloor dx.$$

and therefore, $dx = dx^i \wedge \widehat{dx^i}$.

Note that, for each $1 \le i \le m$, $[\widehat{dx^i}]$ is an orientation on the faces F_i and $F_i + \delta_i e_i$. On the other hand, the orientations with respect to outward pointing vectors are $[(-e_i)\rfloor dx] = -[\widehat{dx^i}]$ for F_i and $[e_i \rfloor dx] = [\widehat{dx^i}]$ for $F_i + \delta_i e_i$.

The integral of a differential $(m-1)$-form

$$\Theta = a_i(x)\widehat{dx^i}.$$

on ∂R using the outward pointing orientations is therefore

$$\begin{aligned}
\int_{\partial R} \Theta &= \sum_{i=1}^{i=m} - \int_{F_i} a_i(x)\, \widehat{dx^i} + \int_{F_i + \delta_i e_i} a_i(x)\, \widehat{dx^i} \\
&= \sum_{i=1}^{i=m} \int_{F_i} (a_i(x + \delta_i e_i) - a_i(x))\, \widehat{dx^i} \\
&= \sum_{i=1}^{i=m} \int_{F_i} \left(\int_{x^i=0}^{x^i=\delta_i} \partial_i a_i(x + x^i e_i)\, dx^i \right) \widehat{dx^i} \\
&= \sum_{i=1}^{i=m} \int_R \partial_i a_i(x)\, dx \\
&= \sum_{i=1}^{i=m} \int_R (\partial_j a_j(x)\, dx^j) \wedge \widehat{dx^i} \\
&= \int_R da_i \wedge \widehat{dx^i}.
\end{aligned}$$

This suggests that naturally associated with the differential $(m-1)$-form Θ is the differential m-form

$$d\Theta = da_i \wedge \widehat{dx^i}.$$

Using this definition, the above shows that

$$\int_{\partial R} \Theta = \int_R d\Theta.$$

7. The exterior derivative of a differential form

Based on the calculation above, it is reasonable to define the exterior derivative of a differential $k-1$-form

$$\Theta = \sum_{1\le i_1,\dots,i_{k-1}\le m} a_{i_1\cdots i_{k-1}} dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}},$$

to be

$$d\Theta = \sum_{1\le i_1,\dots,i_{k-1}\le m} da_{i_1\cdots i_{k-1}} \wedge dx^{i_1}\wedge\cdots\wedge dx^{i_{k-1}},$$

It is, however, necessary to show that this definition is independent of the coordinates used. This is a consequence of the next section.

8. The pullback of the exterior derivative

A crucial property of the exterior derivative is that, given any differential form Θ on an open $O' \subset \mathbb{R}^n$ and a map $\Phi : O \to O'$,

$$\Phi^* d\Theta = d\Phi^*\Theta.$$

We prove this below. For convenience we denote $\Phi(x) = (y^1(x), \dots, y^n)$.

First,

$$\begin{aligned}\Phi^*(df) &= \Phi^*\left(\frac{\partial f}{\partial y^\alpha}\, dy^\alpha\right)\\ &= \frac{\partial f}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^i}\, dx^i\end{aligned}$$

and, by the chain rule,

$$\begin{aligned}d(\Phi^* f) &= d(f(y(x)))\\ &= \frac{\partial f}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^i}\, dx^i\\ &= \Phi^*(df).\end{aligned}$$

Next, if $\theta = df$, then, since partials compute,

$$\begin{aligned}d\theta &= d(df)\\ &= \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta}\, dy^\alpha\wedge dy^\beta\\ &= 0\end{aligned}$$

and therefore

$$\Phi^*(d\theta) = 0.$$

On the other hand,

$$\begin{aligned} d(\Phi^*\theta) &= d(\Phi^* df) \\ &= d\left(\Phi^*\left(\frac{\partial f}{\partial y^\alpha} dy^\alpha\right)\right) \\ &= d\left(\frac{\partial f}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^i}\, dx^i\right) \\ &= \left(\frac{\partial^2 f}{\partial y^\beta \partial y^\alpha}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\alpha}{\partial x^i} + \frac{\partial f}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^j \partial x^i}\right) dx^j \wedge dx^i \\ &= 0. \end{aligned}$$

Finally, given a differential $k-$form

$$\Theta = a_{\alpha_1\cdots\alpha_k}(y)\, dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_k},$$

$$\begin{aligned} \Phi^*(d\Theta) &= \Phi^*(da_{\alpha_1\cdots\alpha_k} \wedge dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_k}) \\ &= (\Phi^* da_{\alpha_1\cdots\alpha_k}) \wedge \Phi^*(dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_k}) \\ &= d(\Phi^* a_{\alpha_1\cdots\alpha_k}) \wedge \Phi^*(dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_k}) \\ &= d((\Phi^* a_{\alpha_1\cdots\alpha_k})\, \Phi^*(dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_k})) \\ &= d(\Phi^*\Theta). \end{aligned}$$

A corollary of this is that the definition of the exterior derivative is invariant under changes of coordinates. It also shows that the exterior derivative of a differential form on a submanifold is independent of the parameterization.

9. Stokes's theorem for a rectangular region

Theorem 1. *Let $m \le n$, $R \subset \mathbb{R}^m$ be a rectangular region, and $S \subset \mathbb{R}^n$ a submanifold with a piecewise smooth boundary oriented by outward vectors, and parameterized by a map $\Phi : R \to \mathbb{R}^n$. Given any differential $(m-1)$-form Θ on an open neigborhood O' of S,*

$$\int_{\partial S} \Theta = \int_S d\Theta.$$

Proof.

$$\int_{\partial S} \Theta = \int_{\Phi(\partial R)} \Theta = \int_{\partial R} \Phi^*\Theta = \int_R d(\Phi^*\Theta) = \int_R \Phi^*(d\Theta) = \int_{\Phi(R)} d\Theta = \int_S d\Theta.$$

□

10. Integration over a triangle

Let $T \subset \mathbb{R}^2$ be the triangle with vertices at $(0,0),(1,0),(0,1)$. The integral of $\Theta = a\,dx + b\,dy$ along the boundary of T oriented counterclockwise is

$$\begin{aligned}
\int_{\partial T} \Theta &= \int_{x=0}^{x=1} a(x,0)\,dx + \int_{x=1}^{x=0} a(x,1-x) - b(x,1-x)\,dx + \int_{y=1}^{y=0} b(0,y)\,dy \\
&= -\int_{x=0}^{x=1} a(x,1-x) - a(x,0)\,dx + \int_{y=0}^{y=1} b(1-y,y) - b(0,y)\,dy \\
&= -\int_{x=0}^{x=1} \left(\int_{y=0}^{y=1-x} \partial_y a(x,y)\,dy \right) dx + \int_{y=0}^{y=1} \left(\int_{x=0}^{x=1-y} \partial_x b(x,y)\,dx \right) dy \\
&= -\int_T \partial_y a(x,y)\,dx \wedge dy + \int_T \partial_x b(x,y)\,dx \wedge dy \\
&= \int_T (\partial_y a(x,y)\,dy + \partial_x a(x,y)\,dx) \wedge dx + (\partial_x b(x,y)\,dx + \partial_y b(x,y)\,dy) \wedge dy \\
&= \int_T da \wedge dx + db \wedge dy.
\end{aligned}$$

11. Integration over a simplex

Let $\Delta \subset \mathbb{R}^m$ be the simplex with vertices at $0, e_1, \dots, e_m$. In other words,

$$\Delta = \{(x^1, \dots, x^m) \;:\; 0 \le x^1, \dots, x^m, x^1 + \cdots + x^m \le 1\}.$$

The boundary of Δ consists of $(n+1)$ faces, given by

$$F_0 = \{(x^1, \dots, x^m) \;:\; x^1 + \cdots + x^m = 1,\ 0 \le x^1, \dots, x^m \le 1\}$$

and, for each $1 \le i \le m$,

$$F_i = \{(x^1, \dots, x^m) \;:\; x^i = 0,\ 0 \le x^1, \dots, x^m, x^1 + \cdots + x^m \le 1\}$$

Corresponding outward vectors are $n_0 = e_1 + \cdots + e_m$ and, for each $1 \le i \le m$, $n_i = -e_i$. As before, denote $dx = dx^1 \wedge \cdots \wedge dx^m$ and

$$\widehat{dx^i} = e_i \lrcorner dx.$$

The integrals below use the orientations $[\widehat{dx^1} + \cdots + \widehat{dx^m}]$ on F_0 and, for each $1 \le i \le m$, $[\widehat{dx^i}]$ on F_i. Note that, if $1 \le j \ne i \le m$, then $\widehat{dx^j}$ restricted to F_i is zero. The integral of a differential $(m-1)$-form

$$\Theta = a_j\,\widehat{dx^j}$$

over $\partial\Delta$ with the orientation induced by outerward vectors is therefore

$$\begin{aligned}
\int_{\partial\Delta} \Theta &= \int_{F_0} a_j\,\widehat{dx^j} - \sum_{i=1}^m \int_{F_i} a_j\,\widehat{dx^j} \\
&= \sum_{i=1}^m \left(\int_{F_0} a_i \widehat{dx^i} - \int_{F_i} a_i\,\widehat{dx^i} \right),
\end{aligned}$$

For each $1 \le i \le m$, the face F_0 can be parameterized by the map $\Phi_i : F_i \to F_0$, where

$$\Phi_i(x^1, \dots, x^m) = (x^1, \dots, x^m) + (1 - x^1 - \cdots - x^m)e_i.$$

It follows that

$$\begin{aligned}
\int_{\partial\Delta} \Theta &= \sum_{i=1}^{m} \int_{F_i} a_i(x + (1 - x^1 - \cdots - x^m)e_i) - a_i(x)\, \widehat{dx^i} \\
&= \sum_{i=1}^{m} \int_{F_i} \left(\int_{x^i=0}^{x^i=1-x^1-\cdots-x^m} \partial_i a_i(x + x_i e_i)\, dx^i \right) \widehat{dx^i} \\
&= \int_{\Delta} (\partial_j a_i(x)\, dx^j) \wedge \widehat{dx^i} \\
&= \int_{\Delta} da_i \wedge \widehat{dx^i}
\end{aligned}$$

We therefore have proved the following theorem.

Theorem 2. *Let $m \leq n$, $\Delta \subset \mathbb{R}^m$ be a simplex, and $S \subset \mathbb{R}^n$ a submanifold with a piecewise smooth boundary oriented by outward vectors with a parameterization $\Phi : \Delta \to \mathbb{R}^n$. Given any differential $(m-1)$-form Θ on an open neigborhood O' of S,*

$$\int_{\partial S} \Theta = \int_S d\Theta.$$

Corollary 3. *If Θ is a differential $(m-1)$-form on a smoothly triangulated m-dimensional manifold M, then*

$$\int_{\partial M} \Theta = \int_M d\Theta.$$