CURVATURE AND THE THEOREMA EGREGIUM OF GAUSS

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In this note, we describe a simple way to define the second fundamental form of a hypersurface in $\mathbb{R}^n$ and use it to prove Gauss's Theorema Egregium, as well as its analogue in higher dimensions.

The basic idea is to approximate a hypersurface, near a given point, by the graph of a quadratic polynomial near its critical point. In particular, given any point p in a hypersurface $S \subset \mathbb{R}^n$, the hypersurface can be positioned so that p is at the origin and S is tangent to the hyperplane $T = \{x^n = 0\}$. It follows that S is locally the graph of a function $f(x^1, \dots, x^{n-1})$ satisfying $f(0) = 0$ and $df(0) = 0$. The second order Taylor expansion of f is therefore

$$f(v) = H(v, v) + o(|v|^2),\ v \in T,$$

where H is a symmetric tensor. The second fundamental form at p is defined to be H. The uniqueness of H implies that it is an extrinsic geometric invariant tensor of S.

The statement of the Theorem Egregium then falls out of a straightforward calculation that seeks to identify a tensor invariant, defined in terms of the second fundamental form, that is independent of the extrinsic isometric embedding. It therefore is an intrinsic geometric invariant of the hypersurface. Another straightforward tensor calculation leads to an intrinsic definition of Gauss curvature for a surface in $\mathbb{R}^3$ and, more generally, the Riemann curvature tensor.

Note that neither the Levi-Civita connection nor the Gauss map is used explicitly below.

The discussion below has a straightforward extension to submanifolds of higher codimension.

1. Rigid motions

Let $e_1, \dots, e_n$ denote the standard basis of $\mathbb{R}^n$. Given a nonzero vector $\nu \in \mathbb{R}^n$, let $\nu^\perp$ denote the linear hyperplane normal to ν.

A *rigid motion* is a map $R\colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$R(x) = p + A(x - p),$$

where $A \in \mathrm{SO}(n)$ and $p \in \mathbb{R}^n$.

2. Definition of a hypersurface

Define a *hypersurface* to be a subset $S \subset \mathbb{R}^n$ such that for each $x \in S$, there exists a linear subspace $T \subset \mathbb{R}^n$ of codimension 1 such that S near x is graph of a smooth function of the plane $x + T$. In other words, there exists neighborhood $N \subset \mathbb{R}^n$ of 0 and a smooth function $f : N \cap T \to \mathbb{R}$ such that

$$S \cap (x + N) = \{x + v + f(v)u \ :\ v \in T \cap N\}, \tag{1}$$

where u is a nonzero vector normal to T and $f(0) = 0$.

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3. Tangent space

For each $x \in S$, there is a unique choice of T, which will be denoted $T_x S$ and called the *tangent plane to S at x*, such that the corresponding function f satisfies $f(0) = 0$ and $df(0) = 0$. If S is orientable, then there exists a smooth map $u : S \to S^{n-1}$, where $u(x)$ is normal to $T_x S$. This is called the *Gauss map*. We will not use the Gauss map below.

4. The second fundamental form

For each $x \in S$, let f be the function be the one in (1), with $T = T_x S$. The *second fundamental form* of S at x is defined to be the Hessian $H(x)$ of f at 0. The Hessian $H(x)$ is a symmetric tensor on $T_x S$, where, for any $v \in T_x S$,

$$H(x)(v, v) = \left.\frac{d^2}{dt^2}\right|_{t=0} f(tv).$$

Since f is uniquely determined by S and $x \in S$, so is H. It is therefore a geometric invariant of the surface S. Since it depends on the embedding of S, it is an extrinsic invariant.

Given $x \in S$, we can apply a rigid motion to S so that $x = 0$, $u(x) = e_n$, and therefore, in a neighborhood of $0 \in \mathbb{R}^n$,

$$S = \{(x', f(x')\},$$

where $f(0) = 0$ and $df(0) = 0$. For each $v = v^1 e_1 + \cdots + v^{n-1} e_{n-1} \in T_0 S (= e_n^\perp)$,

$$H(0)(v, v) = v^i v^j H(0)(e_i, e_j) = H_{ij} v^i v^j, \text{ where } H_{ij} = H_{ji} = H(0)(e_i, e_j).$$

5. Isometric hypersurfaces

Let T denote $e_n^\perp$.

Two hypersurfaces S and $\widehat{S}$ are *isometric* if there exists a smooth diffeomorphism

$$\Phi : S \to \widehat{S}$$

such that the length of any smooth curve $C \subset S$ is equal to the length of the curve $\Phi(C) \subset \widehat{S}$.

Given $x \in S$, let $\hat{x} = \Phi(x) \in \widehat{S}$. Applying rigid motions to both S and $\widehat{S}$, we can assume that that $\hat{x} = x = 0$ and the respective tangent hyperplanes are $T_{\hat{x}}\widehat{S} = T_x S = T$. This in turn defines uniquely the functions $f : T \to \mathbb{R}$, whose graph near 0 is S, and $\hat{f} : T \to \mathbb{R}$, whose graph near 0 is $\widehat{S}$.

Moreover, there is a diffeomorphism $\phi : T \to T$ such that, for each $v \in T$,

$$\Phi(v + f(v)e_n) = \phi(v) + \hat{f}(\phi(v))e_n.$$

The diffeomorphism $\Phi : S \to \widehat{S}$ preserves the lengths of all curves if and only if

$$\partial_i \hat{y} \cdot \partial_j \hat{y} = \partial_i y \cdot \partial_j y,$$

where, for each $v \in T$,

$$\begin{aligned} \hat{y}(v) &= \phi^k(v) + \hat{f}(\phi(v))e_n \\ y(v) &= v + f(v)e_n, \end{aligned}$$

for each $v \in T$. In other words,

$$\partial_i \phi \cdot \partial_j \phi + \partial_p \hat{f} \partial_i \phi^p \partial_q \hat{f} \partial_j \phi^q = \delta_{ij} + \partial_i f \partial_j f. \tag{2}$$

Note that

$$\partial_i f(0) = \partial_i \hat{f}(0) = 0$$
$$\partial_i \phi^j(0) = \delta_i^j,$$

for all $1 \leq i, j \leq n-1$. Therefore, if we differentiate (2) and evaluate at $v = 0$, we get

$$\partial^2_{ik}\phi^j + \partial^2_{jk}\phi^i = 0.$$

This implies that, at $v = 0$,

$$\partial^2_{ik}\phi^j = -\partial^2_{jk}\phi^i = \partial^2_{ji}\phi^k = -\partial^2_{ik}\phi^j, \tag{3}$$

and therefore

$$\partial^2_{jk}\phi^i(0) = 0, \text{ for all } 1 \leq i, j, k \leq n-1.$$

If we differentiate (2) again, evaluate at $x = 0$, and cycle through the indices i, j, k, l, we get

$$\partial^3_{ikl}\phi^j + \partial^3_{jkl}\phi^i + \partial^2_{ik}\hat{f}\partial^2_{jl}\hat{f} + \partial^2_{il}\hat{f}\partial^2_{jk}\hat{f} = \partial^2_{ik}f\partial^2_{jl}f + \partial^2_{il}f\partial^2_{jk}f \tag{4}$$
$$\partial^3_{jli}\phi^k + \partial^3_{kli}\phi^j + \partial^2_{jl}\hat{f}\partial^2_{ki}\hat{f} + \partial^2_{ji}\hat{f}\partial^2_{kl}\hat{f} = \partial^2_{jl}f\partial^2_{ki}f + \partial^2_{ji}f\partial^2_{kl}f \tag{5}$$
$$\partial^3_{kij}\phi^l + \partial^3_{lij}\phi^k + \partial^2_{ki}\hat{f}\partial^2_{lj}\hat{f} + \partial^2_{kj}\hat{f}\partial^2_{li}\hat{f} = \partial^2_{ki}f\partial^2_{lj}f + \partial^2_{kj}f\partial^2_{li}f \tag{6}$$
$$\partial^3_{ljk}\phi^i + \partial^3_{ijk}\phi^l + \partial^2_{lj}\hat{f}\partial^2_{ik}\hat{f} + \partial^2_{lk}\hat{f}\partial^2_{ij}\hat{f} = \partial^2_{lj}f\partial^2_{ik}f + \partial^2_{lk}f\partial^2_{ij}f. \tag{7}$$

Therefore, the equation

$$(4) - (5) + (6) - (7) \tag{8}$$

eliminates ϕ and gives

$$\partial^2_{ik}\hat{f}\partial^2_{jl}\hat{f} - \partial^2_{il}\hat{f}\partial^2_{ik}\hat{f} = \partial^2_{ik}f\partial^2_{jl}f - \partial^2_{il}f\partial^2_{ik}f, \tag{9}$$

Ig follows that, if H and $\widehat{H}$ are the second fundamental forms of S and $\widehat{S}$ at 0, then

$$\widehat{R}_{ijkl} = R_{ijkl}, \tag{10}$$

where

$$R_{ijkl} = H_{ik}H_{jl} - H_{il}H_{jk} \tag{11}$$
$$\widehat{R}_{ijkl} = \widehat{H}_{ik}\widehat{H}_{jl} - \widehat{H}_{il}\widehat{H}_{jk}. \tag{12}$$

This leads to the definition of the tensor R, where, for any $v_1, v_2, v_3, v_4 \in T_xS$,

$$R(v_1, v_2, v_3, v_4) = H(v_1, v_3)H(v_2, v_4) - H(v_1, v_4)H(v_2, v_3).$$

What (10) shows is that the tensor R for a surface S depends only on its intrinsic geometry and not on the isometric embedding. The tensor R is, of course, the Riemann curvature tensor, and the equations (11) are the Gauss equations.

If $n = 2$, then the only nontrivial component of the Riemann curvature tensor is

$$K = R(e_1, e_2, e_1, e_2) = H_{11}H_{22} - H_{12}^2,$$

which is known as the Gauss curvature, and equation (10) is the Theorem Egregium of Gauss.

Example 1. We can show that the standard sphere in $\mathbb{R}^3$ is not locally isometric to the plane $x^3 = 0$ using this. Using the definition of the second fundamental form, we can verify that, for the plane, $H_{ij} = 0$ and therefore $R_{1212} = 0$. On the other hand, for the sphere, $H_{ij} = \delta_{ij}$ and therefore $R_{1212} = 1$. Therefore, the two surfaces are not locally isometric.

6. Tensor identities

Most of the proof above involves only differentiation and straightforward calculations. The only significant steps are the following:

(1) Definition of a hypersurface
(2) If a map between two hypersurfaces preserves lengths of curves, then the map satisfies equation (2).
(3) Most importantly, the tensor calculations done in (3) and (8). These are equivalent to the following tensor identities:

$$(T \otimes S^2T) \cap (\Lambda^2 T \otimes T) = \{0\}$$
$$(T \otimes S^3T) \cap (\Lambda^2 T \otimes \Lambda^2 T) = \{0\}.$$

7. Intrinsic formula for Riemann curvature

The embedding of S as a graph over T is given by $v \in T \mapsto v + f(v)e_n$ and therefore, the Riemannian metric is given by

$$\begin{aligned} g_{ij}(v) &= \partial_i(v + f(v)e_n) \cdot \partial_j(v + f(v)e_n) \\ &= (e_i + \partial_i f(v)e_n) \cdot (e_k + \partial_j f(v)e_n) \\ &= \delta_{ij} + \partial_i f \partial_j f. \end{aligned}$$

Therefore, since $df(0) = 0$, it follows that at $v = 0$,

$$\begin{aligned} \partial_k g_{ij} &= \partial_k(\delta_{ij} + \partial_i f \partial_j f) \\ &= \partial^2_{ki} f \partial_j f + \partial_i f \partial^2_{kj} f \\ &= 0. \\ \partial^2_{ij} g_{kl} &= \partial^2_{ij}(\partial_k f \partial_l f) \\ &= \partial^2_{ik} f \partial^2_{jl} f + \partial^2_{il} f \partial^2_{jk} f. \end{aligned}$$

Permuting the indices, we get

$$\partial^2_{ik} g_{jl} = \partial^2_{ij} f \partial^2_{kl} f + \partial^2_{il} f \partial^2_{jk} f \tag{13}$$
$$\partial^2_{jl} g_{ik} = \partial^2_{ij} f \partial^2_{kl} f + \partial^2_{jk} f \partial^2_{il} f \tag{14}$$
$$\partial^2_{il} g_{jk} = \partial^2_{ij} f \partial^2_{kl} f + \partial^2_{ik} f \partial^2_{jl} f \tag{15}$$
$$\partial^2_{jk} g_{il} = \partial^2_{ij} f \partial^2_{kl} f + \partial^2_{jl} f \partial^2_{ik} f. \tag{16}$$

Therefore, the equation

$$-(13) - (14) + (15) + (16) \tag{17}$$

implies that, at $x = 0$,

$$\begin{aligned} -\partial^2_{ik} g_{jl} - \partial^2_{jl} g_{ik} + \partial^2_{il} g_{jk} + \partial^2_{jk} g_{il} &= -\partial^2_{il} f \partial^2_{jk} f - \partial^2_{jk} f \partial^2_{jl} f + \partial^2_{ik} f \partial^2_{jl} f + \partial^2_{jl} f \partial^2_{ik} f \\ &= 2R_{ijkl}. \end{aligned}$$

This proves, at least for a Riemannian manifold that can be isometrically embedded as a hypersurface in Euclidean space, that the Riemann curvature tensor is an intrinsic geometric invariant.

8. The Riemann curvature of an abstract Riemannian manifold

Let M be a n-manifold with Riemannian metric g. Given local coordinates $x^1, \dots, x^n$, the metric is

$$g = g_{ij}(x)\, dx^i\, dx^j,$$

where $g_{ij}(x) = g(\partial_i, \partial_j)$. The calculations in the previous section suggest the following: Given $p \in M$, let $x = (x^1, \dots, x^n)$ be coordinates such that $x(p) = 0$, $g_{ij}(0) = \delta_{ij}$ and $\partial_k g_{ij}(0) = 0$, for all $1 \le i, j, k \le n$. Then define the Riemann curvature tensor at p to be given by

$$R(v_1, v_2, v_3, v_4) = R_{ijkl} v_1^i v_2^j v_3^k v_4^l,$$

where, at $x = 0$,

$$R_{ijkl} = -\frac{1}{2}(\partial^2_{ik} g_{jl} + \partial^2_{jl} g_{ik} - \partial^2_{il} g_{jk} - \partial^2_{jk} g_{il}).$$

To prove that this is a well defined tensor, it suffices to prove the following lemmas, which have straightforward proofs.

Lemma 1. *Let g be a smooth Riemannian metric on an n-manifold M. For each $p \in M$, there exist local coordinates $x = (x^1, \dots, x^n)$ such that $x(p) = 0$ and $g = g_{ij}\, dx^i\, dx^j$, where, for every $1 \le i, j, k \le n$,*

$$\begin{aligned} g_{ij}(0) &= \delta_{ij} \\ \partial_k g_{ij}(0) &= 0. \end{aligned}$$

Remark. Exponential or normal coordinates satisfy Lemma 1. However, a direct proof of Lemma 1 is a lot simpler than the construction of exponential coordinates.

Lemma 2. *Let M and N be smooth n-manifolds, h be a Riemannian metric on N, $\Phi : M \to N$ a smooth map, and $g = \Phi^* h$. Let $x = (x^1, \dots, x^n)$ be local coordinates on a neighborhood of $p \in M$ and $y = (y^1, \dots, y^n)$ be local coordinates on a neighborhood of $\Phi(p) \in N$ such that*

$$\begin{aligned} x(p) &= 0 \\ y(\Phi(p)) &= 0 \\ \frac{\partial y^i}{\partial x^j}(0) &= \delta^i_j \\ h_{ij}(0) &= \delta_{ij} \\ \partial_k h_{ij}(0) &= 0. \end{aligned}$$

Then at $x = y = 0$,

$$\begin{aligned} g_{ij}(0) &= \delta_{ij} \\ \partial_k g_{ij}(0) &= 0 \\ \partial^2_{ik} g_{jl} + \partial^2_{jl} g_{ik} - \partial^2_{il} g_{jk} - \partial^2_{jk} g_{il} &= \partial^2_{ik} h_{jl} + \partial^2_{jl} h_{ik} - \partial^2_{il} h_{jk} - \partial^2_{jk} h_{il}. \end{aligned}$$

The proofs of these lemmas can be found in *A simple way to discover the Riemann curvature tensor*.

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