A SIMPLE PROOF OF HERON'S FORMULA FOR THE AREA OF A TRIANGLE

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I learned following proof of Heron's formula from Daniel Rokhsar.

Theorem 1. *The area of a triangle with side lengths a, b, c is equal to*

$$A(a,b,c) = \sqrt{s(s-a)(s-b)(s-c)}, \tag{1}$$

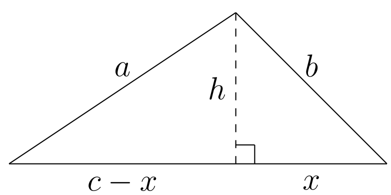
where

$$s = \frac{a+b+c}{2}.$$

Proof. First, observe that the domain of A is the open set

$$\{(a,b,c) \ : \ a,b,c > 0,\ b+c-a > 0,\ c+a-b > 0,\ a+b-c > 0\}.$$

Next, we show that A^2 is a polynomial function of a, b, c.



The following equations hold:

$$\begin{aligned} h^2 + (c-x)^2 &= a^2 \\ h^2 + x^2 &= b^2. \end{aligned}$$

Subtracting these equations, we get

$$\begin{aligned} b^2 - a^2 &= x^2 - (c-x)^2 \\ &= c(2x - c) \end{aligned}$$

and therefore

$$cx = \frac{1}{2}(b^2 + c^2 - a^2).$$

It follows that

$$\begin{aligned} A^2 &= \frac{1}{4}c^2h^2 \\ &= \frac{1}{4}c^2(b^2 - x^2) \\ &= \frac{1}{4}(b^2c^2 - (cx)^2) \\ &= \frac{1}{4}\left(b^2c^2 - \frac{1}{4}(b^2 + c^2 - a^2)^2\right). \end{aligned} \tag{2}$$

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It follows that A^2 is a fourth degree homogeneous polynomial in a, b, c. Since the function A^2 is a symmetric function of a, b on an open subset on $\mathbb{R}^3$, it must also be a symmetric polynomial.

It is now a straightforward but tedious algebraic calculation to show that (2) is equivalent to (1). Below is an alternative proof that avoids this calculation.

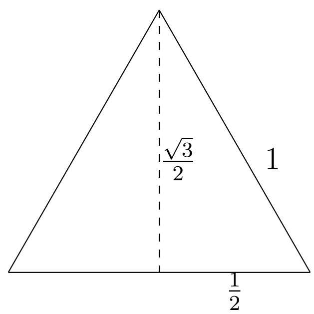
If $a = b + c$, $b = c + a$, or $c = a + b$, then $A(a, b, c) = 0$. It follows that A^2 can be factored as

$$A^2 = f(a, b, c)(b + c - a)(c + a - b)(a + b - c).$$

where f is a symmetric polynomial of degree 1. The only such polynomials are of the form

$$f(a, b, c) = C(a + b + c),$$

where C is a constant. To determine the value of C, we can use any triangle whose area is easy to compute. Consider the equilateral triangle with sides of length 1,



On one hand,

$$A = \frac{1}{2}ch = \frac{\sqrt{3}}{4},$$

and, on the other hand,

$$A^2 = C(a + b + c)(b + c - a)(c + a - b)(a + b - c) = 3C.$$

Therefore,

$$C = \frac{1}{16},$$

and

$$A = \sqrt{\left(\frac{a + b + c}{2}\right)\left(\frac{b + c - a}{2}\right)\left(\frac{c + a - b}{2}\right)\left(\frac{a + b - c}{2}\right)}.$$

□