HOLONOMY IS CURVATURE

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Let E be a vector bundle over a smooth manifold M and ∇ a connection on E. The curvature of the connection is the section Ω of $\bigwedge^2 T^*M \otimes \mathrm{Aut}(E)$ such that

$$\Omega(X,Y)e = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})e \in E_x, \tag{1}$$

for any $x \in M$, $X, Y \in T_xM$, $e \in E_x$.

Given a smooth curve $c : [0,1] \to M$, the parallel transport of $e \in E_{c(0)}$ along c is defined to be the section $f : [0,1] \to E$ such that the following hold for each $t \in [0,1]$:

$$\begin{aligned} f(t) &\in E_{c(t)} \\ f(0) &= e \\ \nabla_T f(t) &= 0, \end{aligned}$$

where $T = \partial_t$. Denote $P_c e = f(1)$.

Let $c : [0,1] \to M$ be a C^1 null-homotopic curve based at x. There exists a C^1 map $C : [0,1] \times [0,1] \to M$ satisfying the following for each $0 \le s, t \le 1$:

$$\begin{aligned} C(0,t) &= x \\ C(1,t) &= c(t) \\ C(s,0) &= x \\ C(s,1) &= x. \end{aligned}$$

Given $e_x \in E_x$, let $e : [0,1] \times [0,1] \to E$ be C^2 section of C^*E satisfying the following for all $0 \le s, t \le 1$:

$$\begin{aligned} e(s,t) &\in E_{C(s,t)} \\ e(s,0) &= e_x \\ \nabla_T e(1,t) &= 0 \\ \nabla_S e(s,t) &= 0, \end{aligned}$$

where $S = \partial_s$ and $T = \partial_t$. In particular,

$$e(s,1) = P_c e_x.$$

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The approach shown here is inspired by the writings of Eschenburg, Heintze, Jost, Karcher.

Let E^* be the dual vector bundle of E. Given $\varepsilon_x \in E^*_x$, let $\varepsilon : [0,1] \times [0,1] \to E^*$ satisfy the following for all $0 \le s, t \le 1$:

$$\begin{aligned}
\varepsilon(s,t) &\in E^*_{C(s,t)} \\
\varepsilon(0,t) &= \varepsilon_x \\
\varepsilon(s,0) &= \varepsilon_x \\
\varepsilon(s,1) &= \varepsilon_x \\
\nabla_S \varepsilon(s,t) &= 0.
\end{aligned}$$

It follows that

$$\nabla_T \varepsilon(0,t) = 0.$$

Lemma 1.

$$\langle \varepsilon_x, P_c e_x - e_x \rangle = \int_{[0,1]\times[0,1]} \langle \varepsilon(s,t), C^*\Omega e \rangle.$$

Proof.

$$\begin{aligned}
\langle \varepsilon_x, P_c e_x - e_x \rangle &= \langle \varepsilon(0,1), e(0,1) \rangle - \langle \varepsilon(0,0), e(0,0) \rangle \\
&= \int_{t=0}^{t=1} \partial_t(\langle \varepsilon(0,t), e(0,t) \rangle)\, dt \\
&= \int_{t=0}^{t=1} \langle \varepsilon, \nabla_T e(0,t) \rangle\, dt \\
&= \int_{t=0}^{t=1} \left[\langle \varepsilon, \nabla_T e(1,t) \rangle - \int_{s=0}^{s=1} \partial_s(\langle \varepsilon, \nabla_T e(s,t) \rangle)\, ds \right] dt \\
&= -\int_{t=0}^{t=1} \int_{s=0}^{s=1} \langle \varepsilon, \nabla_S \nabla_T e(s,t) \rangle\, ds\, dt \\
&= \int_{t=0}^{t=1} \int_{s=0}^{s=1} \langle \varepsilon, \Omega(C_* T, C_* S) e(s,t) \rangle\, ds\, dt \\
&= \int_{[0,1]\times[0,1]} \langle \varepsilon, C^*\Omega e \rangle.
\end{aligned}$$

□

A corollary of this is the Ambrose-Singer theorem [1]. An elegant presentation of the above can be found in lecture notes of Werner Ballman[2].

References

[1] W. Ambrose and I. M. Singer. *A theorem on holonomy*. Trans. Amer. Math. Soc. 75 (1953), pp. 428–443. DOI: 10.2307/1990721.

[2] W. Ballman. *Vector Bundles and Connections*. 2002. URL: http://people.mpim-bonn.mpg.de/hwbllmnn/archiv/conncurv1999.pdf (visited on 03/2002).